TopQuants Newsletter

Volume 4, Issue 2 November 2016

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Editorial

Dear Reader,

the TopQuants team presents the second issue of our 2016 newsletter series.

As always we cordially invite all readers to contact us with your ideas and submissions. Anything that is relevant to our quant audience, is more than welcome!

The current issue will kick off with a summary of the Spring Event held at KPMG in May this year. This event focused on complexity theory and financial regulation. Our guests, who co-authored an article in the famous Science magazine, shared with the TopQuants' society why the traditional economic theory has not been able to explain, or even predict, the near collapse of the financial system and its long-lasting effects on the global economy. If you want to learn more, but missed the event, do not forget to visit page 2.

The second article is a summary of the Quant Careers 2016 event, at which four former students Sjoerd van Bakel (Vrije Universiteit), Jordi Rustige (Vrije Universiteit), Kevin van der wees (Quant Trainee at APG Asset Management, Vrije Universiteit) and Jasper Faber (Rabobank, Vrije Universiteit) battled it out against each other in order to decide who is the winner of the Best Quant Finance Thesis Award 2016. See page 3 to find out who claimed the title of the best upcoming quant in the Netherlands in 2016.

The next article is an extended summary of one of the speakers at the last Autumn Event 2015 Lech Grzelak (ING Bank), who explains how to "fix" arbitrage-generating volatility parametrizations and shows how to determine an arbitrage-free density based on an arbitrage-generating volatility parameterizations in particular Hagan's formula.

The final article was delivered to us by Roald Waaijer, Michiel Hopman and Bauke Maarse from Deloitte, the host of the upcoming Autumn Event 2016. They discuss interest rate averaging as a new optionality within mortgage products, which was recently introduced. Their article offers an explanation of this development and evaluates the impact on risk management and pricing.

Two upcoming quant events are the Winter School on Mathematical Finance that will take place on January 23-25, 2017 in Lunteren, and the upcoming Maths Olympiad for Corporates, which will be hosted by Optiver on the 27th of January. You can read more about these in the Upcoming Events section on the last page of this newsletter.

Moreover, we are delighted to announce that APG will host the Spring Event in April or May 2017. Stay tuned for more information.

The first newsletter of 2017 will also contain coverage of the upcoming Autumn Event, kindly hosted by Deloitte.

We hope you will enjoy reading this newsletter and we look forward to seeing you at the upcoming TopQuants event(s).

On behalf of the TopQuants team,

Marcin Rybacki

TopQuants Spring Event - 2016

Complexity theory and financial regulation

The recent publication of **Prof. dr. Cars Hommes** (Economic Dynamics, University of Amsterdam), **prof. dr. Hans Heesterbeek** (Theoretical Epidemiology, Leiden University) and **dr. Diego Garlaschelli** (Theoretical Physics, Leiden University) discusses recent insights and techniques that offer potential for better monitoring and management of highly connected and interdependent financial systems. TopQuants were honoured to have the three co-authors of the **Science*** paper present at the event.

From the perspective of the financial crisis the topic seemed very up-to-date and attracted over hundred financial professionals in that warm, spring, afternoon. They all gathered at the KPMG headquarters in Amsterdam on the 12th of May 2016.

Ted van der Aalst (KPMG), the TopQuants facilitator of the event started with a warm word of welcome. After that, Albert Röell (Chief Executive Officer KPMG in the Netherlands) welcomed the audience on behalf of the host and sponsor of the Spring event. He recalled an amusing anecdote about a physicist, a chemist, and a quant who were stranded on a desert island with no implements and a can of food. The physicist and the chemist each devised an ingenious mechanism for getting the can open, whereas the quant merely said, "Assume we have a can opener"!

The main part of the programme combined three presentations and a panel discussion. Hans Heesterbeek was the first speaker. He described the complexity theory from the point of view of the research area he represented - biology. He explained that, similar to other complex systems such as climate or ecosystems, also in the economy one can observe that before a major transition, there is often a gradual and unnoticed loss of resilience. A small trigger can collapse the entire system like a domino. Hans showed that there are means to quantify and detect tipping points,. Those could be for instance: rising correlation between nodes in a network and rising temporal variance, correlation and skewedness of fluctuation patterns. These indicators, first predicted mathematically, were subsequently assessed in real systems, including living systems.

The second presentation was given by Diego Garlaschelli, who talked about complexity theory from the econophysics perspective. He showed, based on the study of the Dutch interbank network, that standard analysis using a homogeneous network model could only lead to late detection of the 2008 crisis, however when using a more realistic and heterogeneous network model, one could identify an early warning signal even 3 years before the crisis. Diego emphasized that estimating systemic risks relies on very specific data on the financial network. He admitted that it was difficult to capture the full picture due to the fact that business interactions between banks are often hidden because of confidentiality issues. However, as he noted, there are tools being developed to reconstruct networks from partial information and to estimate systemic risk.

In the third talk Cars Hommes gave a short lecture on how laboratory experiments with human subjects can provide empirical validation of individual decision rules of agents, their interactions, and emergent macro behaviour. The experiments conducted by Cars revealed that economic systems may deviate significantly from rational efficient equilibrium at both individual and aggregate levels. Such feature of positive feedback systems may lead to continuous deviations of prices from equilibrium and emergence of speculation-driven bubbles and crashes.

After all three presentations, all speakers participated in a lively panel discussion moderated by Roger Lord (Cardano). One of the questions concerned the lowering of interest rates by the ECB and whether this was a positive signal. Hans replied that too much regulation and standardization of the system would not make it more stable. Biology shows that the important thing is a proper distribution of weights, like predators and the circulation biomass. Another question related to the fall of Lehman Brothers. Diego answered that there is no evidence that a big player is more dangerous for the network, it is about changing the equilibrium state.

After the discussion, everybody had ample opportunity for informal discussions and networking during a complimentary buffet dinner and drinks session.

TopQuants would like to thank:
– all speakers for their contributions,
– KPMG for hosting and sponsoring the event,
– all participants for attending.

** Stefano Battiston, J. Doyne Farmer , Andreas Flache, Diego Garlaschelli, Andrew G. Haldane,Hans Heesterbeek, Cars Hommes, Carlo Jaeger, Robert May, Marten Scheffer,* ***Complexity theory and financial regulation, Science*** *19 Feb 2016: Vol. 351, Issue 6275, pp. 818-819*

Quant Careers 2016 Minisymposium

(Diederik Fokkema, Elangovan Krishnan EY)

On the 4th of November, 2016 TopQuants, in cooperation with EY and Quants@VU, organised the third edition of the **Best Quant Finance Thesis Award** at EY office in Amsterdam. At first candidates had to submit their theses, of which the top 4 competitors were selected to pitch at the symposium for the victory and the title of the best young quant thesis in 2016, in front of the jury and the invited guests. The event was opened by **Diederik Fokkema** (EY), the president of TopQuants, and **Svetlana Borovkova** (Associate Professor at Vrije Universiteit).

The growing popularity of the contest (30 submissions this year, with 3 of them from female students) among the students was emphasised, as well as very high quality of the theses, which made the choice extremely difficult for the jury. Svetlana gave some interesting statistics about the theses submitted. Compared to previous years, the length of the thesis decreased but the quality remained intact. The disappointing fact was the small representation of the female students in the competition. The contestants represented all top Dutch universities with quantitative programs. Also the range of research topics was very wide – from the valuation of exotic options, XVA, credit risk to risk management in pensions as well as financial econometrics. The theses of the three finalists will be considered as The Netherlands' submission to the European Quant Awards 2016. The jury consisted of: **Roger Lord** (Cardano), **Maurits Bakker** (PwC), **Matteo Michielon** (ABN AMRO), **Natalia Migal** (Leaseplan), and **Frank Pardoel** (RiskQuest).

The opening lecture was given by guest speaker, **Ingrid Gacci**, Head of Accounting, Operations and Compliance at Intesa Sanpaolo Bank, Amsterdam. The content of the speech was about the risk of money laundering, a practical approach of dealing with this risk based on scenario analysis, and a comparison in this respect between the Netherlands and Italy. Ingrid showed that according to the estimations even up to USD 2.1 trillion comes from criminal proceeds. This sums up to approximately 3.6% of the global GDP. What is more, according to the OECD, the total value of bribes paid worldwide can reach USD 1 trillion per annum.

After Ingrid's presentation it was time for the four pitching of the top contenders. The first one was given by **Sjoerd van Bakel** (Vrije Universiteit), whose thesis covered the topic of CVA and DVA on Conic Commodity Options. The next pitching was from **Jordi Rustige** (Vrije Universiteit) on the topic of Perceived financial networks & systemic risk. Both presentations were made by pre-recorded video on display with the presentation of slides in parallel. The third presentation was by **Kevin van der Wees** (Quant Trainee at APG Asset Management, Vrije Universiteit), on a generalized approach for quantifying model risk at financial institutions and the final presentation by **Jasper Faber** (Rabobank, Vrije Universiteit) on pricing collateralized contracts using a novel distribution based approach.

As the jury was busy in selecting the best thesis for the year 2016, the second guest speaker **Esther Mollema**, the founder of leadership consultancy firm Direction was introduced by Svetlana. The brilliant presentation was made on about why risk is your best friend in work and life. The presentation has a short fun exercise, but the conclusion was on choosing the right job with enough diversity to get challenged, which puts you on the ladder of success and improvement.

The final phase of the event was completed with announcement of winner of the best thesis award for 2016. The winner was judged to be **Jordi Rustige**, and prof. Svetlana Borovkova took the honour of accepting the award (cash prize of 2500, sponsored by EY) on student's behalf. The second prize went to **Jasper Faber**, the third prize to **Sjoerd van Bakel**, and the remaining slot (fourth best thesis) went to **Kevin van der Wees**. The event concluded with informal drinks.

Interest averaging impacts risk management practices of financial institutions

Interest rate averaging is a new optionality within mortgage products, which was recently introduced. This article offers an explanation of this development and evaluates the impact on risk management and pricing.

Due to a declining trend of consumer rates in the Dutch mortgage market over the past years, interest rate averaging has gained an increased regulatory and media attention. This year, interest rate averaging is or will be introduced by multiple mortgage providers. As interest rate averaging is a new phenomenon, the impact on risk management and capital is widely unknown. Public perception is that interest rate averaging appears to be a goldmine for banks (FD, 2/2016). This article provides more insight into market practices on interest rate averaging as applied by Dutch Mortgage providers.

What is interest rate averaging?

Customers who borrowed money against a fixed rate some years ago pay a much higher interest rate than the current mortgage rate, due to the decline in interest rates. Interest rate averaging can be seen as resettlement of an existing mortgage loan where the client immediately profits from the lower mortgage rate, while not having to pay an upfront penalty. Instead of paying the prepayment penalty, the penalty is spread out over the new fixed interest period of the newly settled mortgage. This can be advantageous for customers who want to decrease their monthly coupon payment. Aspects that influence whether interest rate averaging is beneficial for the client include the level of the rate, the type of mortgage, and the remaining fixed interest period of the mortgage. The market (i.e. clients, regulators, the government, competitors) forces financial institutions to take a position on offering interest rate averaging.

Regulatory restrictions

Regulation poses a number of conditions on the method used for interest rate averaging. These restrictions include the following:

1) In the Mortgage Credit Directive it is stated that the prepayment penalty should not exceed financial loss of the creditor. Hence, no add-ons above financial loss can be incorporated in the new client coupon;

2) Generally prepayment due to relocation does not result in a prepayment penalty. However, relocating after interest rate averaging can be advantageous for customers, since the penalty that is spread out over the new interest fixed period is only paid until the relocation. Introducing a relocation penalty can therefore be considered. The Code of Conduct for Mortgage Loans includes a condition on maximum compensation for prepayments due to relocation, namely the maximum of 4 months of interest on the prepaid amount and 3% of the prepayment amount;

Moreover in political debates, the following technical points regarding the methodology were mentioned about interest rate averaging [3]:

- Calculation of the net present value (NPV) should be in the interest of the customer;
- The total spread (add-on) should not exceed the maximum of 20bps (to allow for tax deduction).

> "Interest rate averaging: goldmine for banks"
> **Financieele Dagblad (translated), 11/02/2016 [1]**

Market research

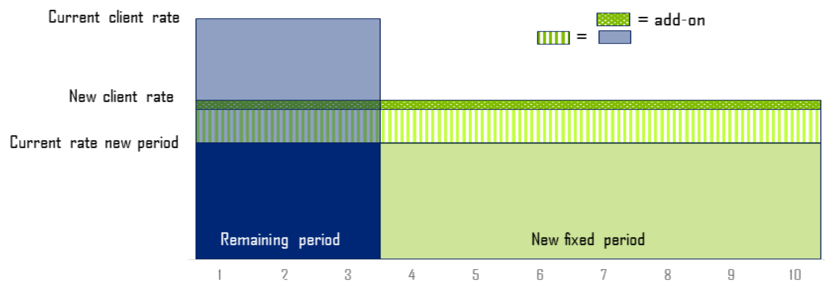
The calculation methods used by mortgage providers combined with the conditions applied result in a wide range of interest rate averaging approaches. Deloitte Financial Risk Management performed a benchmark study. On a high level three following approaches are applied:

- **Simple weighted average:** weighted average of the historical rate and the actual rate for the new period, where the weights are based on the old and new interest fixed period (see illustration above). On top of this rate an additional penalty margin of 0.2% can be included;
- **Adjusted weighted average**: similar to the weighted average but with a difference reference

Illustration – Example of interest rate averaging

One way to calculate the rate after averaging is by calculating the weighted average of the client rate and the current rate for the new period. Given that the client rate is 5% and the client has 3 years remaining and the current rate for a 10-year period is 2.6% then the rate after averaging is 3.32%[1]. On top of this rate an additional spread can be added. Note that this is just one way to calculate the rate after interest rate averaging.

Figure - 1: Graphical illustration of interest rate averaging



$$\frac{2.6\% \cdot 10 + (5\% - 2.6\%) \cdot 3}{10} = 3.32\%$$

"The total upward potential, measured over the new interest period, is small and in certain cases even negative."

MoneyView conclusion on the effect of interest rate averaging for customers (translated), 10/8/2016 [2]

rate. The current mortgage rate for the remaining period is used as a reference rates instead of the rate for the new period;

- **Net present value**: a more complex method where the prepayment penalty is spread over the new fixed interest period using a net present value approach.

Impact of Interest rate averaging

Interest rate averaging has a significant impact on various aspects within risk management and pricing. It will impact among others Internal Liquidity Adequacy Assessment Process (ILAAP), Interest Rate Risk in the Banking Book (IRRBB), market value, Solvency Capital Requirement (SCR) ratio, securitization and pricing.

ILAAP

In the ILAAP, repayments on mortgages are used to define the liquidity risk. These repayments are based on the expected behavioral repayments of the notional. Behavioral notional payments of a mortgage are earlier than contractual due to prepayments. These prepayments occur due to various reasons and will be affected by introducing interest rate averaging. By introducing interest rate averaging external refinancing is less attractive compared to internal refinancing, therefore customers are more likely to stay with their bank. This results in a reduction of the prepayment rates due to the introduction of interest rate averaging.

IRRBB

Similar to ILAAP, also the interest rate risk measures will change due to changes in prepayment speed. Next to the prepayment speed also the client coupon changes. Since, interest rate risk considers the interest and notional cash flows over a specific period. These cash flows are affected by interest rate averaging in two ways:

- The interest typical prepayment rates change (prepayment changes the term of the interest contract);
- The coupon changes (e.g. used to measure interest income).

Overall, interest rate averaging results in less refinancing with a prepayment penalty, due to extension of the current contract using an adjusted rate.

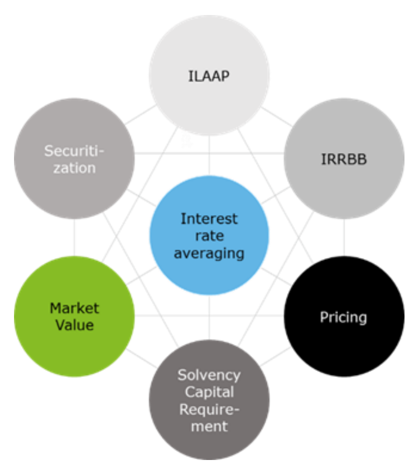


Figure 2:
Aspects affected by interest rate averaging.

Market value

Interest rate averaging affects the market value due to a lower coupon and a longer fixed interest period. The impact on the market value is highly dependent on the method used to define the client coupon after interest rate averaging. Due to the following two countervailing effects the market value can either increase or decrease: A longer interest contract increases market value due to the margin earned and the lower coupon decreases market value.

Solvency Capital Requirement (SCR) ratio

Interest rate averaging affects the SCR ratio (own funds divided by SCR). Both the own funds and the SCR are affected. The change in market value leads to a change in own funds. The SCR is affected due to longer interest rate contracts, which in general result in a higher risk and therefore a higher SCR.

Securitization

Financial institutions may have securitized part of their mortgage portfolio. These securitizations may opt restrictions towards introducing new optionalities as interest rate averaging. Securitized assets have been bought with certain assumptions on optionalities. By introducing additional optionalities these restrictions may be broken and for example a buy-back may be necessary.

Pricing

Interest rate averaging changes the pricing of mortgages due to changes in optionalities and credit risk. Examples of optionalities are the prepayment option, take-along (meeneem) option (the option to keep the mortgage contract and corresponding coupon after relocating) and interest rate averaging itself. Ideally, each of these optionalities should be included in the pricing of a mortgage, which increases the client rate. This increase will (partly) be offset by lower credit risk. Credit risk decreases as the lower client coupon reduces monthly payments for the client and therefore reduces the probability on a default.

To summarize, interest rate averaging affects a wide range of metrics and can therefore have a significant impact on risk management and pricing.

Deloitte FRM Services

Deloitte Financial Risk Management (FRM) can support in selecting an approach, defining impact and incorporating interest rate averaging in pricing and risk management.

Topics that Deloitte FRM can support include:
- Analysis of different interest rate averaging approaches;
- Defining the areas of impact, based on the approach and type of institution;
- Calculate the impact in the current situation and the impact under different interest rate environments;
- Determine how the models, methods and underlying assumptions shall be adjusted in order to incorporate interest rate averaging in pricing and risk measurement systems.

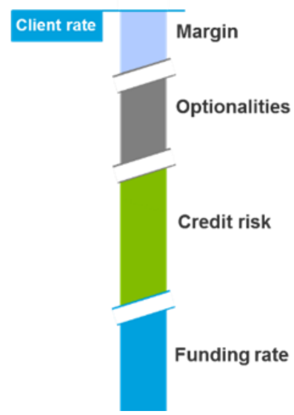


Figure 3: Illustration of pricing of mortgages.

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[3] Antwoorden op kamervragen rentemiddeling, Rijksoverheid, 2015.

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On how to "fix" arbitrage-generating volatility parametrizations*

by Lech A. Grzelak (ING Bank - Quantitative Analytics)
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ING BANK

Abstract

We propose a method for determining an arbitrage-free density based on an arbitrage-generating volatility parameterizations in particular Hagan's formula[1][4]. Our method is based on the stochastic collocation method [7]. Analytic European option prices are available and the implied volatilities stay very close to those initially obtained from the parameterization. The proposed method is very fast and straightforward to implement as it only involves 1D Lagrange interpolation and inversion of a linear system of equations. The method is generic and may be applied to other variants or other models that generate arbitrage.

1. Introduction

When handling a large number of market volatility quotes it is natural to express them in terms of some parametric form so that a whole range of strikes can be explained by only a few parameters. Once the parametric equation is given, one can instantly obtain volatilities by evaluating the parametric function.

The Stochastic Alpha Beta Rho (SABR) model from [4] is described by the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}S(t) &= \sigma(t)S^{\beta}(t)\mathrm{d}W_1(t), \quad S(t_0)=S_0, \\ \mathrm{d}\sigma(t) &= \gamma\sigma(t)[\rho\mathrm{d}W_1(t)+\sqrt{1-\rho^2}\mathrm{d}W_2(t)], \\ \sigma(t_0) &= \alpha \end{aligned}$$

where $S(t)$ is the forward rate, S_0 the initial forward rate, σ represents the stochastic volatility, and with the parameters ρ, β, γ, α denoting, respectively, the correlation, the skew, the volatility of volatility (vol-vol) and the overall level of the volatility parameters. The model is popular in the financial industry because of the availability of an analytic asymptotic implied volatility formula (derived with the help of perturbation theory).

This implied volatility parameterization formula is often used in the financial industry for expressing the market quotes, even for options with expiry times of twenty years or more. It is however a well-known fact that the accuracy of this so-called Hagan formula deteriorates with time and so the occurrence of implied densities giving rise to arbitrage opportunities increases as the option expiry times increase.

Our approach differs from the ones available in the literature as we do not seek for a better analytic expression for the implied volatilities from the SABR model but we project, by means of a coordinate transformation, the survival probability onto another stochastic variable which leads to an arbitrage-free density. The concept can be expressed as follows: assuming that Y is a random variable corresponding to the model used for parameterizing the volatilities and X is a known random variable, e.g. a Gaussian, we determine a coordinate transformation $y = g(x)$ for which European call prices (and their implied volatilities) are identical, i.e.:

$$\int (y-K_i)^+ f_Y(y)\mathrm{d}y = \int (g(x)-K_i)^+ f_X(x)\mathrm{d}x,$$
$$i \in \{1,\ldots,N\}.$$

When the coordinate transformation is known, one can use it for pricing any plain vanilla product while benefiting from the fact that the density $f_X(x)$ specified is free of arbitrage. In short, the proposed method can be used to approximate a random variable Y by a polynomial based on normal variables (or another variable), i.e.,

$$Y \sim a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots,$$

where the coefficients α_1, α_2,... are inferred from a mapping and solving a small system of equations.

Our preferred method for determining the mapping relies on the stochastic collocation method [7].

2. Basics of stochastic collocation and implied density

Let us start with some intuition behind the collocation method. The method is developed to approximate an *expensive* to *compute* stochastic variable Y by means of a *cheap* variable X. An approximation is made based on the inversion of the CDF of Y at only a small set of col-

* *For the complete article see [2]*

[1] *We use the wording "Hagan's formula" as an abbreviation for the Hagan-Kumar-Leśniewski-Woodward model.*

location points, being the zeros of an orthogonal polynomial.

The stochastic collocation method can be used to approximate a cumulative distribution function (CDF). As any CDF is uniformly distributed, we have $F_Y(Y) \equiv F_X(X)$. This equality in distribution does not imply that X and Y are equal in distribution, but only that the CDFs follow the same uniform distribution. From the representation above, realizations of Y, y_n, and X, x_n, are connected via the following inversion relation,

$$y_n = F_Y^{-1}(F_X(x_n)). \tag{2.1}$$

The objective is to determine an alternative relation which does not require many "expensive" inversions $[F_Y]^{-1}(\bullet)$ for all samples of X. The task is to find a function $g(\bullet)=[F_Y]^{-1}(F_X(\bullet))$ such that

$$F_X(x) = F_Y(g(x)) \qquad Y \stackrel{\mathrm{d}}{=} g(X),$$

where evaluations of function $g(\bullet)$ do not require the inversions $[F_Y]^{-1}(\bullet)$. With a mapping $g(\bullet)$ determined, the CDFs $F_X(x)$ and $F_Y(g(x))$ are not only equal in distributional sense but also element-wise.

Sampling from random variable Y can be decomposed into sampling from a cheap random variable X and a transformation to Y via $g(\bullet)$ i.e., $y_n = g(x_n)$. It is important to choose $g(\bullet)$ for a simple, basic function.

An efficient method for sampling from variable Y in terms of variable X is obtained by defining $g(\bullet)$ to be a *polynomial expansion*, i.e.

$$y_n \approx g_N(x_n) = \sum_{i=1}^{N} y_i \ell_i(x_n),$$

$$\ell_i(x_n) = \prod_{j=1, i\neq j}^{N} \frac{x_n - \bar{x}_j}{\bar{x}_i - \bar{x}_j}, \tag{2.2}$$

where x_n is a sample from X and $\bar{x}_i$ *and* $\bar{x}_j$ are so-called *collocation points*, y_i is the exact evaluation at collocation point $\bar{x}_i$ in (2.1), i.e. $y_i = [F_Y]^{-1}(F_X(\bar{x}_i))$ in (2.2). A particular choice for the collocation points $\bar{x}_i$ is discussed in [3].

2.1 Implied density

In this section, we discuss how to determine an arbitrage-free density based on $f_Y(\bullet)$ of Y. Typically, the implied density has problems around 0 where the absorption property is not properly handled in the formula[2]. The density deteriorates in a region near zero (see Fig. 1, upper picture). We will map Y onto a random variable X, such that the mapping procedure takes place in those regions where the density of Y is properly defined.

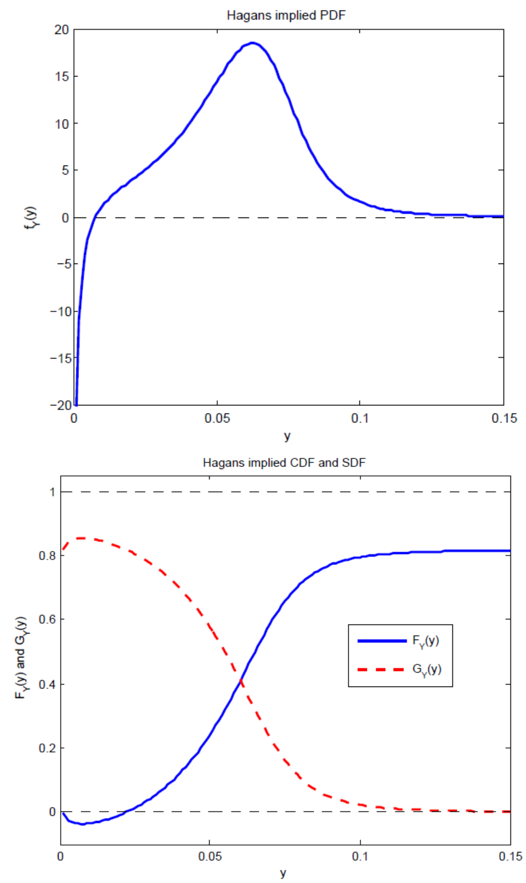


Figure 1: Parameter values $\beta = 0.5$, $\alpha = 0.05$, $\rho = -0.7$, $\gamma = 0.4$, $F(t_0) = 0.05$ and T=7. Up: probability density, with deterioration near zero; down: corresponding CDF (blue) and SDF (dotted red).

The representation in (2.2) with $y_i = [F_Y]^{-1}(F_X(\bar{x}_i))$ is not yet well-suited. The main problem comes from the fact that the implied CDF does not have the natural [0,1] bounds, as shown in Figure 1 (right-side picture). Since the density can become negative, CDF $F_Y(y)$ exhibits an upper bound which is less than one.

Since $F_Y(\bullet)$ is not well-defined the inversion $[F_Y]^{-1}(F_X(\bar{x}_i))$ will give us incorrect mapping points. Figure 1 shows however that, although $F_Y(y)$ does not have proper upper and lower bounds, the *survival distribution function* (SDF), defined by means of the European call options, $V_{call}(t_0,K)$, with strike K, as

$$G_Y(y) = 1 - \int_{-\infty}^{y} f_Y(x)\mathrm{d}x = \int_{y}^{+\infty} f_Y(x)\mathrm{d}x = \boxed{-\frac{\partial V_{\mathrm{call}}(t_0,K)}{\partial K}\Big|_{K=y}}, \tag{2.3}$$

[2] By using the so-called "shifted variant model", i.e. Based on $S + \theta$, see [5], the instability at zero shifts towards $S = \theta$.

has a natural limit value of 0 for $y \to \infty$. By focusing on the survival distribution $G_Y(\bullet)$, we can make use of the collocation mapping which is given by:

$$y \approx g_N(x) = \sum_{i=1}^{N} G_Y^{-1}(G_X(\bar{x}_i))\ell_i(x) =: \sum_{i=1}^{N} y_i \ell_i(x), \tag{2.4}$$

Because the inversion $[G_Y]^{-1}(G_X(\bar{x}_i))$ is only well-defined in the part of $G_Y(\bullet)$ which is monotone, we set specific values g_{min} and g_{max} and choose the collocation points so that $G_X(\bar{x}_i) > g_{min}$. In other words, the values of g_{min} and g_{max} determine the range at which we can be confident about the quality of the mapping between the two variables.

When the limits g_{min} and g_{max} are prescribed, the collocation method maps the survival probability of Y onto a survival probability based on a polynomial of X. In Figure 2 an interpolation (right upper figure) takes place at the variable level (X,Y), which is typically rather smooth and almost linear (see [3] for a more detailed discussion). In order to apply Lagrange interpolation between the nodes $(\bar{x}_i, y_i)$ it is important to make use of optimal collocation points $\bar{x}_i$ ensuring that the polynomial has certain optimality properties, and avoiding any oscillations to occur.

In the next subsection we will discuss the relation between the densities of Y and X.

2.2 Recovery of the PDF and pricing options

By the definition of function $g(x)$, we have:

$$\begin{aligned} y = g(x) &\stackrel{\text{def}}{=} G_Y^{-1}(G_X(x)), \\ G_X(x) = G_Y(g(x)) &=: G_Y(y). \end{aligned} \tag{2.5}$$

Differentiating (2.5) w.r.t. x results in:

$$\frac{\mathrm{d}G_X(x)}{\mathrm{d}x} = -f_X(x),$$

$$\frac{\mathrm{d}G_Y(g(x))}{\mathrm{d}x} = -f_Y(g(x))\frac{\mathrm{d}g(x)}{\mathrm{d}x},$$

and the relation between the densities is therefore given by:

$$\begin{aligned} f_Y(g(x)) &= f_X(x)\left(\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right)^{-1} \\ &\approx f_X(x)\left(\frac{\mathrm{d}g_N(x)}{\mathrm{d}x}\right)^{-1}, \end{aligned} \tag{2.6}$$

with $g_N(x)$ as in (2.4) and derivative $dg_N(x)/dx$ is known analytically, see below in (2.7).

Remark: (Efficient evaluation of $x = g^{-1}(y)$). *Since mapping $y = g(x)$ is bijective and $g(x)$ is strictly increasing, so is $g^{-1}(y)$. This implies that the arguments x can be obtained by the inverse interpolation [6] of $g(x)$ against y, which can be done at essentially no cost.*

With $g_N(x)$ the Lagrange polynomial, its derivative reads:

$$\frac{\mathrm{d}g_N(x)}{\mathrm{d}x} = \sum_{i=1}^{N} y_i \frac{\mathrm{d}\ell_i(x)}{\mathrm{d}x} = \sum_{i=1}^{N} y_i \ell_i(x) \sum_{j=1, j\neq i}^{N} \frac{1}{x - \bar{x}_j}, \tag{2.7}$$

and the density can be further simplified to:

$$f_Y(g(x)) \approx f_X(x)\left(\sum_{i=1}^{N} y_i \ell_i(x) \sum_{j=1, j\neq i}^{N} \frac{1}{x - \bar{x}_j}\right)^{-1}. \tag{2.8}$$

Using the results above, we can price European-style payoffs highly efficiently, as:

$$\begin{aligned} V(t_0, y_0) &= \mathbb{E}\left[V(T, Y(T))|\mathcal{F}(t_0)\right] \\ &= \int_{G_X^{-1}(1)}^{G_X^{-1}(0)} V(T, g(x)) f_Y(g(x)) \frac{\mathrm{d}g(x)}{\mathrm{d}x}\mathrm{d}x, \end{aligned} \tag{2.9}$$

which, by Equation (2.6) and the approximation $g(x) \approx g_N(x)$, gives:

$$\begin{aligned} V(t_0, y_0) &= \int_{G_X^{-1}(1)}^{G_X^{-1}(0)} V(T, g(x)) f_X(x)\mathrm{d}x \\ &\approx \int_{G_X^{-1}(1)}^{G_X^{-1}(0)} V(T, g_N(x)) f_X(x)\mathrm{d}x. \end{aligned} \tag{2.10}$$

Although the pricing of options is generally done numerically, by integrating expression in (2.10), European put and call option prices are known analytically when X is a Gaussian variable (see Section 2.3).

2.3 Analytic European option prices for normal collocation variable

Before we give the analytic expression for European option prices, we recall the formulas for the moments of a truncated normal distribution.

Result 2.1 (The moments for a truncated univariate normal distribution). *Let $X \sim N(0,1)$ and $a \in (-\infty, \infty)$, then the expression for the moments $m_i := \mathbf{E}[X^i \mid X > a]$, reads*

$$m_i = (i-1)m_{i-2} + \frac{a^{i-1} f_{\mathcal{N}(0,1)}(a)}{1 - F_{\mathcal{N}(0,1)}(a)}, \; i = 1, \ldots, \tag{2.11}$$

with $m_{-1} = 0$, $m_0 = 1$ *and* $f_{N(0,1)}(x)$ *and* $F_{N(0,1)}(x)$ *the standard normal probability and cumulative distribution functions, respectively.*

In the following lemma we show that European option prices under $g_N(x)$ with $X \sim N(0,1)$ are known analytically.

Lemma 2.1 (European call option prices). *With the collocation random variable* $X \sim N(0,1)$ *for* $g_N(X)$*, European call prices are analytically available, and given by:*

$$V_{call}(t_0, K) \approx \int_{G_X^{-1}(1)}^{G_X^{-1}(0)} (g_N(x) - K)^+ \, f_{\mathcal{N}(0,1)}(x)\mathrm{d}x$$

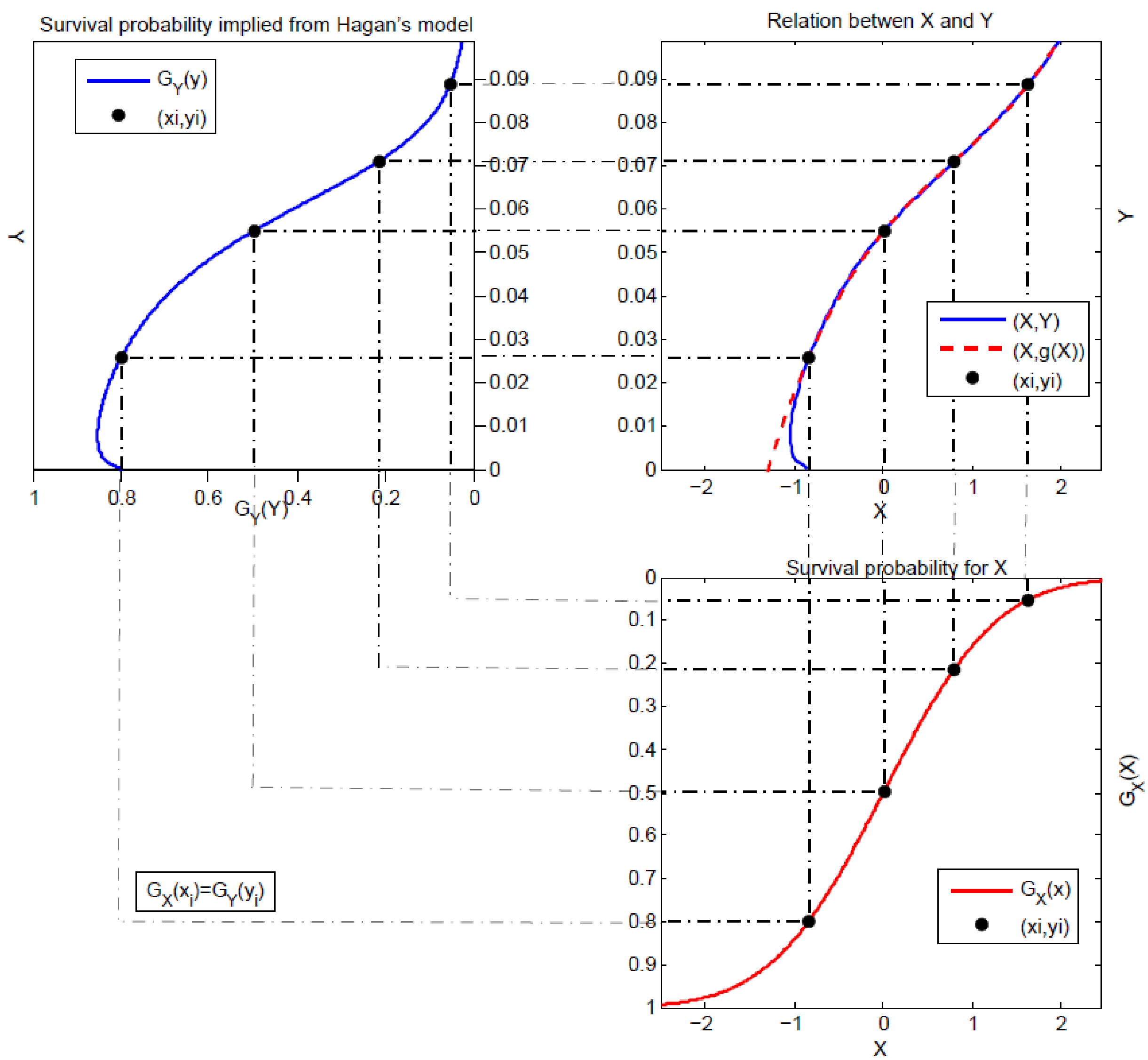
$$= G_{\mathcal{N}(0,1)}(c_K)\left[\sum_{i=0}^{N-1} a_i \mathbb{E}[X^i | X > c_K] - K\right],$$

with $c_K = [g_N]^{-1}(K)$, $G_{N(0,1)}(c_K) = 1 - F_{N(0,1)}(c_K)$, $\mathbf{E}[X^i | X > c_K]$ *the moments of the truncated normal variable, given Result 2.1 and where* a_i, $i \in \{0,\ldots,N-1\}$, are (constant) coefficients obtained by inverting Vandermonde matrix, V, in the matrix equation $\mathbf{Va} = \mathbf{y}$. The k'th row of the matrix V is given by

$$(1, \bar{x}_k^1, \bar{x}_k^2, \ldots, \bar{x}_k^{N-1}),$$

with $\bar{x}_i$ the predetermined collocation points.
Proof. See [2].

Figure 2: Illustration of the mappings of *Y* and *X~N(0,1)* with a polynomial $g_N(X)$.



By the put-call parity and the lemma above, put option prices are also available in closed form. Moreover, with analytic European option prices the calculation of the corresponding implied volatility is a trivial exercise.

3. Market examples

We test our method with SABR parameters that are well known from the literature. Different parameter combinations are presented in Table 1, where the option expiry varies from 1y to 15y. In the experiments we show the generated densities and the corresponding implied volatilities. In all experiments we consider four collocation points that are determined based on strikes $y_i = K_i$, as presented in [2]. The method employs these collocation points to reproduce the implied volatilities from the market.

Table 1: Model parameters chosen in experiments.

Parameters:	β	α (ATM)	ρ (Corr)	γ (vol-vol)	$F(t_0)$	T
Set I as in [1]	0.6	0.25	-0.8	0.3	1	10
Set II as in [5]	0.25	0.35	-0.1	1	1	1

In Figures 3 and 4 the obtained survival probabilities (SDF) and the implied volatilities are presented.

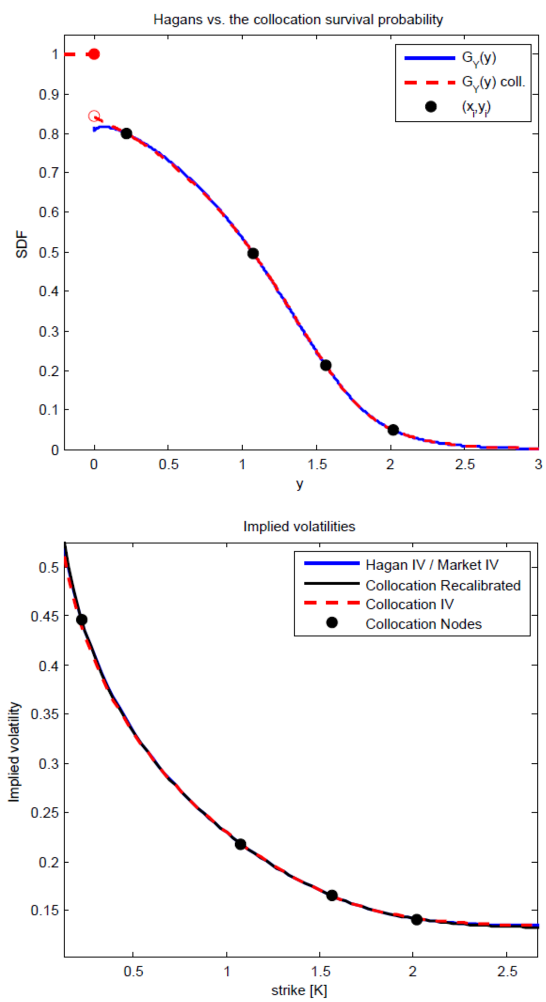


Figure 3: Survival probabilities and implied volatilities for Set I (as given in [1]) with and without re-calibration. The experiment was performed with $g_{min} = 0.05$ and $g_{max} = 0.8$. $G_Y(y)$ stands for the implied survival probability while "$G_Y(y)coll$" indicates the survival probability obtained from the collocation method.

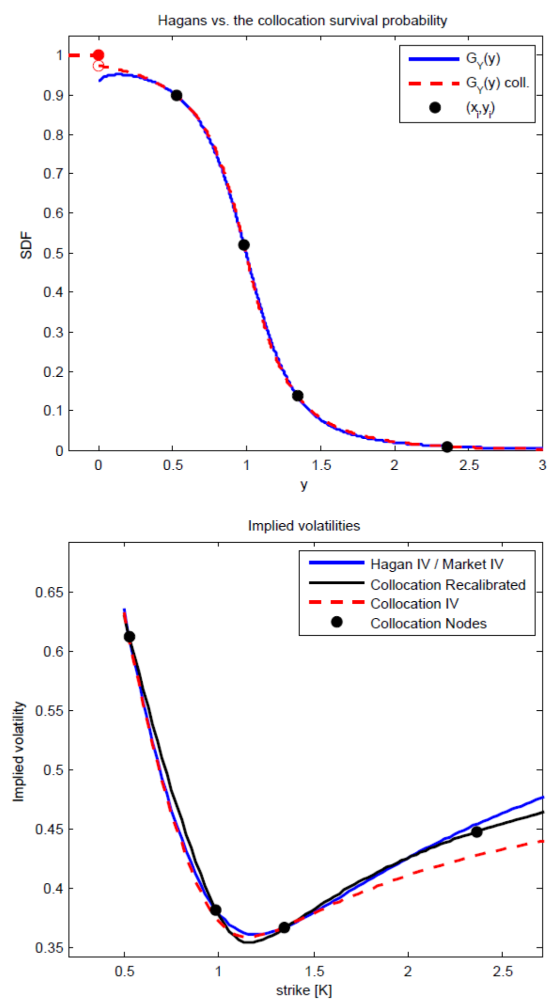


Figure 4: Survival Survival probabilities and implied volatilities for Set II (as given in [5]) with and without re-calibration. The experiment was performed with $g_{min} = 0.01$ and $g_{max} = 0.9$.

In all two cases the SDFs from the collocation method are as desired, i.e., they are monotone and their limits are 0 and 1. The resulting implied volatilities are not all very close to the market values. This can be improved by performing the re-calibration step. In all two cases the re-calibration results in an almost perfect implied volatility match at the collocation points. We also note that the tail asymptotics and the level of curvature and skewness were preserved by the stochastic collocation method. With as few as four collocation points, in all examples, a wide range of implied volatility shapes were generated.

Because the re-calibration step requires only local optimization iterations, it is very fast. The full projection and calibration procedure takes less than 0.1 second for all the cases considered.

4. Conclusions

We have presented an application of the stochastic collocation method for obtaining an arbitrage-free density based on Hagan's formula. Our method relies on the availability of a survival distribution function, which is not necessarily well-defined on the whole domain, and it is projected onto a Gaussian variable. The method presented gives implied volatilities in accordance with those obtained by the model, however, in some cases a re-calibration step is required to guarantee a perfect fit. The method is easy to implement as it only relies on Lagrange interpolation and the solution of a linear system of equations.

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Upcoming Quant Events

TopQuants Autumn Event - Current Topics in Modelling

The workshop will take place on Thursday 24 November at 15:00 and will be held at Deloitte's Amsterdam offices (The Edge). At the event, we will host inspiring parallel sessions with lively debates on a wide range of topics. Talks this year cover topics such as machine learning, adjoint differentiation, credit, capital and liquidity risk modelling, but also how market makers use models, and even a talk on how to build an effective quant team. The first parallel session this year consists of the following talks:

- Jakob Bosma (ING) – Duelling policies: why systemic risk taxation can fail
- Peter den Iseger (ABN AMRO) – New developments in risk modelling: affine models
- Rik Ghijsels (IMC) – Models @ IMC
- Christian Kahl (FINCAD) – Standardised approach for CVA: pricing and risk with adjoint differentiation
- Sjors van der Stelt (Deloitte) – Comprehensive capital analysis and review (CCAR)

whereas the second parallel session will give a platform to the following speakers:

- Ioannis Anagnostou (ING) – Machine learning algorithms for risk management in trading activities
- Roger Holtus (Kleynen Consultants) and Antoon Pelsser (Kleynen Consultants, Maastricht University) – Things a TopQuant rarely talks about
- Asma Khedher (UvA) – Model risk and robustness of quadratic hedging strategies
- Stratos Nikolakakis (ABN AMRO) – Liquidity risk modelling in CSA derivative portfolio(s)
- Florian Reuter (Deloitte) and Mart Stokkers (Rabobank) – Capital framework of defaulted assets under IRB

For more details and abstracts please visit our TopQuants website. The event will be concluded with the usual complimentary drinks and walking dinner, during which all participants can enjoy plenty of networking opportunities.

Winter School on Mathematical Finance

In recent years, the mathematical theory associated with financial risk management and the pricing of contingent claims has been a highly active field of research. The area has established itself as one of the most vigorously growing branches of applied mathematics. Model-based analysis of contracts and portfolios has become a standard in the finance industry, and the number of academic institutions offering curricula in financial mathematics has increased rapidly. In this context, the winter school on Mathematical Finance that will take place on January 23–25, 2017 in Lunteren aims at providing a meeting place for participants both from industry and from academia. The program provides ample opportunity for discussion.

The special topics of the 16th winter school are Polynomial models, and Market imperfections. These are the subjects of minicourses that will be taught by two distinguished speakers: Professor Damir Filipovic (EPFL Lausanne) and Professor Jan Kallsen (Christian-Albrechts-Universitaet zu Kiel). Additionally there will be three one-hour lectures by Professors Erhan Bayraktar (University of Michigan), Thorsten Schmidt (University of Freiburg) and Wim Schoutens (KU Leuven). Thirty-minute lectures on recent research in the Netherlands will be presented by Anne Balter (Tilburg University), Qian Feng (CWI, Amsterdam), Rutger-Jan Lange (VU Amsterdam) and Anton van der Stoep (Rabobank).
For more details please visit the website of the event.

Math Olympiad for Corporates

Another edition of the Maths Olympiad for Corporates (Wiskunde Olympiade voor Bedrijven) will be hosted next year by Optiver. Save the date on the 27th of January 2017. More details will follow in a separate mailing, or can be obtained via email to: mathsolympiad@optiver.com