Direct versus Iterated Cumulative Variance Forecasting

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Motivation and Contribution

- Correct assessment of future (multi-period) volatility is important.
  $\Rightarrow$ How to construct multi-period volatility forecasts?
- Autoregressive modelling: direct h-step ahead forecast is found to be more robust to misspecification than iterated forecast (Marcellino et al. (2006)).
- Can we find a similar effect for volatility forecasting, in particular when the true volatility displays long memory?

This paper:
- Proposes a parameter estimation method for traditional volatility models based on daily data to construct cumulative volatility forecasts.
- Assesses the distance between the proposed and the standard QML estimators' probability limits to conclude about model misspecification using the metric of Hausman (1978).

Estimation Methods

Consider **GARCH(1,1)** model specified for daily log-returns:

$$r_t = \sigma_t z_t, \quad z_t|\mathcal{F}_{t-1} \sim \text{i.i.d.}(0,1) \qquad \sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2, \tag{1}$$

where $\sigma_t^2 = \mathrm{var}[r_t|\mathcal{F}_{t-1}]$ and $\boldsymbol{\theta} = (\omega, \alpha, \beta)'$.

Standard QML

Maximise Gaussian quasi-likelihood function over $\boldsymbol{\theta}$:

$$L_T^d(\boldsymbol{\theta}) = -\frac{1}{2}\sum_{t=1}^{T}\left(\ln(\sigma_t^2(\boldsymbol{\theta})) + \frac{r_t^2}{\sigma_t^2(\boldsymbol{\theta})}\right), \tag{2}$$

where the quasi-ML estimator is $\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}\in\Theta} L_T^d(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^p$. Define $r_{t,h} = \sum_{j=0}^{h-1} r_{t+j}$ and $\mathcal{F}_t = \sigma\{r_s : s \leq t\}$ to use $\hat{\boldsymbol{\theta}}$ to construct *iterated* cumulative return volatility forecast:

$$\widehat{\mathrm{var}}[r_{t,h}|\mathcal{F}_{t-1}] = \sum_{j=0}^{h-1}\hat{\sigma}_{t+j|t-1}^2 =: v_h(\sigma_t^2(\hat{\boldsymbol{\theta}}); \hat{\boldsymbol{\theta}}), \tag{3}$$

where

$$\hat{\sigma}_{t+j|t-1}^2 = \bar{\sigma}^2 + (\hat{\alpha} + \hat{\beta})^j(\hat{\sigma}_{t|t-1}^2 - \bar{\sigma}^2), \quad \hat{\sigma}_{t|t-1}^2 = \hat{\sigma}_t^2 \quad \text{and} \quad \bar{\sigma}^2 = \frac{\hat{\omega}}{1 - \hat{\alpha} - \hat{\beta}}. \tag{4}$$

$\hat{\boldsymbol{\theta}}$ affects $\widehat{\mathrm{var}}[r_{t,h}|\mathcal{F}_{t-1}]$ in two ways:

- In the **functional form** of the relation between the h-step and 1-step volatility forecast (determined by $\hat{\alpha} + \hat{\beta} \Rightarrow$ cannot handle long memory well for large h).
- In the **filtering** of the 1-step volatility forecast.

Our horizon-tuned QML

Maximise Gaussian quasi-likelihood function based on $T - h + 1$ overlapping h-day cumulative log-returns over $\boldsymbol{\theta}$:

$$L_T^o(\boldsymbol{\theta}) = -\frac{1}{2}\sum_{t=1}^{T-h+1}\left(\ln v_h(\sigma_t^2(\boldsymbol{\theta}); \boldsymbol{\theta}) + \frac{r_{t,h}^2}{v_h(\sigma_t^2(\boldsymbol{\theta}); \boldsymbol{\theta})}\right), \tag{5}$$

where the horizon-tuned quasi-ML estimator is $\tilde{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}\in\Theta} L_T^o(\boldsymbol{\theta})$, and $r_{t,h}^2$ denotes the sum of squared returns $r_{t,h}^2 = r_t^2 + \ldots + r_{t+h-1}^2$. Similarly, $\tilde{\boldsymbol{\theta}}$ can be used to construct $v_h(\sigma_t^2(\tilde{\boldsymbol{\theta}}); \tilde{\boldsymbol{\theta}})$, *direct* form of cumulative return volatility forecasting.

GARCH specification for Realised Measures

- Upon availability of daily realised measures RM_t, $RM_{t,h} = \sum_{j=0}^{h-1} RM_{t+j}$ is a better measurement of $\sum_{j=0}^{h-1}\sigma_{t+j|t-1}^2$ rather than $r_{t,h}^2$ (Andersen and Bollerslev (1998)).
- If the **signal is stronger**, we could get a more accurate cumulative variance *direct* forecast by maximising

$$L_T^{rm} = -\frac{1}{2}\sum_{t=1}^{T-h+1}\left(\ln v_h(\sigma_t^2(\boldsymbol{\theta}); \boldsymbol{\theta}) + \frac{RM_{t,h}}{v_h(\sigma_t^2(\boldsymbol{\theta}); \boldsymbol{\theta})}\right). \tag{6}$$

$\Rightarrow$ Specify a model for $\sigma_t^2(\boldsymbol{\theta})$ using realised measures to exploit high-frequency information.

Note: The model for realised measures can also be used to construct *iterated* forecast.

- Consider **multiplicative error model** (MEM) of Engle and Gallo (2006):

$$RM_t = \sigma_t^2 u_t, \quad \mathbb{E}[u_t|\mathcal{F}_{t-1}] = 1 \qquad \sigma_t^2 = \omega + \alpha RM_{t-1} + \beta\sigma_{t-1}^2, \tag{7}$$

where $\mathbb{E}[RM_t|\mathcal{F}_{t-1}] = \sigma_t^2$ and $\mathcal{F}_t = \sigma\{RM_s : s \leq t\}$.

- Using the law of iterated expectations (LIE), one can show that

$$\mathbb{E}[RM_{t+j}|\mathcal{F}_{t-1}] = \bar{\sigma}^2 + (\alpha + \beta)^j(\sigma_{t|t-1}^2 - \bar{\sigma}^2) \tag{8}$$

- To accommodate flexibility of our horizon-tuned QML estimator, we also consider a **component GARCH** model of Engle and Lee (1999).

Note: Both GARCH and cGARCH models are used for return and realised measure distributions.

Hausman-type specification test

- The estimators $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ have been obtained by using the first-order conditions

$$\sum_{t=1}^{T} s_t^d(\hat{\boldsymbol{\theta}}) = 0, \qquad \sum_{t=1}^{T-h+1} s_t^o(\tilde{\boldsymbol{\theta}}) = 0. \tag{9}$$

- The scores s_t^d and s_t^o evaluated at the true value $\boldsymbol{\theta}_0$ satisfy the condition $\mathbb{E}[s_t^d] = \mathbb{E}[s_t^o] = 0$.
- We define $s_t^o = 0$ for $t = T - h + 2, \ldots, T$.
- The joint asymptotic distribution of the two estimators:

$$\sqrt{T}\begin{pmatrix}\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \\ \tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\end{pmatrix} \xrightarrow{d} \mathcal{N}\left(0, H_0^{-1}C_0H_0^{-1}\right), \quad H_0 = \begin{bmatrix}H_0^d & 0 \\ 0 & H_0^o\end{bmatrix}, \quad C_0 = \begin{bmatrix}C_0^{dd} & C_0^{do} \\ C_0^{od} & C_0^{oo}\end{bmatrix} \tag{10}$$

- To consistently estimate C_0, we use the Newey-West (Bartlett kernel) type of HAC estimator with the lag truncation parameter l.
- **Hausman test** considers the testing situation:

$$\mathbb{H}_0: \ \mathrm{plim}(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}) = 0, \tag{11}$$

$$\mathbb{H}_1: \ \mathrm{plim}(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}) \neq 0, \tag{12}$$

with the **Hausman test statistic** defined as

$$H_T = T(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}})'\hat{V}^{-1}(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}), \tag{13}$$

where $\hat{V}$ denotes a consistent estimator of the asymptotic variance-covariance matrix of $\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}\right)$.

- Under $\mathbb{H}_0$: $H_T \xrightarrow{d} \chi_p^2$, where $p = \mathrm{rank}(\hat{V})$, under $\mathbb{H}_1: H_T \xrightarrow{p} \infty$.

Monte Carlo

DGP for Size: GARCH model with $\omega = 0.0156, \alpha = 0.1181, \beta = 0.8693, \nu = 7$ (df).

	$T = 5{,}000$					$T = 20{,}000$				
	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$
Nominal size	$l = 160$	$l = 100$	$l = 11$	$l = 22$	$l = 33$	$l = 160$	$l = 100$	$l = 11$	$l = 22$	$l = 33$
0.2	0.169	0.14	0.136	0.204	0.263	0.139	0.128	0.122	0.172	0.216
0.1	0.086	0.082	0.08	0.136	0.172	0.07	0.055	0.057	0.101	0.138
0.05	0.044	0.048	0.044	0.085	0.119	0.038	0.029	0.026	0.055	0.084
0.01	0.015	0.016	0.016	0.041	0.062	0.01	0.003	0.002	0.014	0.028

Table 1: Empirical size: percentage of rejections of $\mathbb{H}_0$ in Hausman test for GARCH model when DGP is GARCH. l indicates the number of lags in the Newey-West estimator of C depending on horizon h in our horizon-tuned QML estimator. $1{,}000$ simulations is run.

DGP for Power: FIGARCH(1,d,1) model

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \left[1 - \beta L - (1 - \phi L)(1 - L)^d\right] r_t^2.$$

Number of lags in the Newey-West estimator of C is according to Table 1.

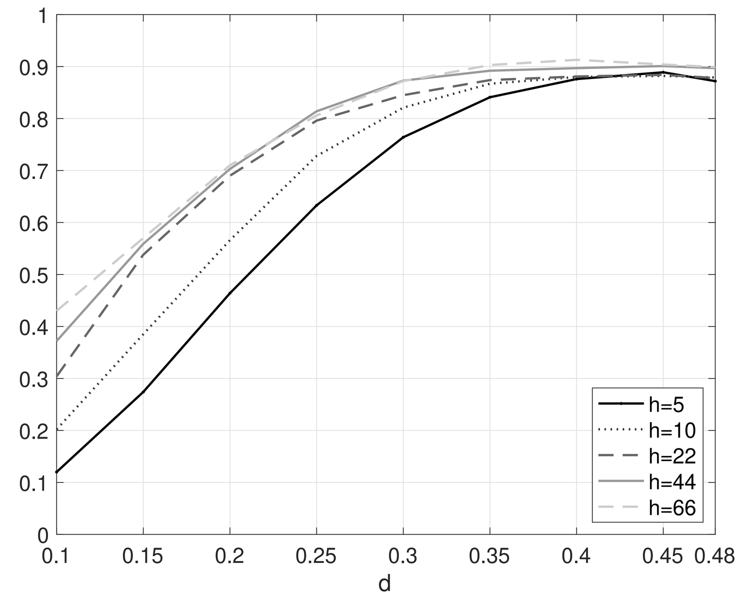

Figure 1: Power curves: percentage of rejections of $\mathbb{H}_0$ in Hausman test for GARCH model when DGP is FIGARCH(0,d,0) at 5% level. $1{,}000$ simulations is run.

	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$
$d = 0.1$	0.146	0.188	0.220	0.244	0.248
$d = 0.3$	0.186	0.248	0.295	0.319	0.328
$d = 0.48$	0.241	0.327	0.396	0.433	0.447

Table 2: Mean Euclidean distance between $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ for each h (for a subset of d values).

Empirical analysis

Data: 10 individual stocks constituting the DJIA index (Gorgi et al. (2019)).
Sample: open-to-close returns and realised kernel for $T = 2515$ trading days (January 2, 2001 – December 31, 2010).
Loss function to compare competing forecasts – QLIKE with realised kernel.

- Both **in-sample** and **out-of-sample**, our horizon-tuned QML estimator yields more accurate forecasts for **realised measure models**.
- **Out-of-sample** gains are more pronounced during **non-crisis periods** (December 2006 – July 2007, January 2010 – December 2010), as concluded from Diebold-Mariano test.

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