Arbitrage-free volatility parameterizations with stochastic collocation

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The Collocation method

In this presentation we will discuss two numerical problems:

- Suppose we consider a problem of sampling of 1.000.000 samples from a variable Y for which the inverse CDF is unknown analytically. A standard procedure is to invert numerically 1.000.000 times the CDF:

$$y_i = F_Y^{-1}(u_i), \quad u_i \sim \mathcal{U}([0,1]).$$

  Problem: How to obtain 1.000.000 samples from Y by using only **a few** inversions $F_Y^{-1}(u_i)$?

- It is common to use parametric representations of the implied volatilities [Hagan et al., 2002, Gatheral and Jacquier, 2013]. These representations often violate the arbitrage assumptions.

  Problem: How to fix an arbitrage-generating volatility parametrization?

The Collocation method

- The collocation technique originates from the field of Uncertainty Quantification [Babuška et al., 2007, Xiu and Hesthaven, 2005].
- Here [Grzelak et al., 2014] we will show how to use the collocation method to approximate any random variable Y by a polynomial of normals (or any other variable), i.e.,

$$Y \sim a_0 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots =: Z,$$

such that:

a) $\mathbb{E}[Y^n] = \mathbb{E}[Z^n], \quad \forall n \in \mathbb{N}$
b) The CDFs of Y and an approximation agree at the so-called collocation points.
c) **No optimization technique will be used!**

- By using of the collocation method to a "well-behaved" region of Hagan's [Hagan et al., 2002] implied CDF we are able to fix the butterfly arbitrage present in the model. Details can be found in [Grzelak and Oosterlee, 2014].

The Collocation method

- Let us consider an "expensive" random variable Y and a "cheap" variable X.
- As any CDF is uniformly distributed we have

$$F_Y(Y) \stackrel{\mathrm{d}}{=} F_X(X).$$

- From the representation above we see that samples of Y, y_n, and X, x_n, are related via the following inversion,

$$y_n = F_Y^{-1}(F_X(x_n)). \quad (1)$$

- Obviously, the sampling via (1) is considered expensive as for each *cheap* realization of X one needs to calculate the inverse of the *expensive* CDF of Y.
- The main objective here is to find an alternative relation which does not require inversions $F_Y^{-1}(\cdot)$ for all samples of X.

The Collocation method

- The task is thus to find a function

$$g(\cdot) = F_Y^{-1}(F_X(\cdot))$$

such that,

$$\boxed{F_X(x) = F_Y(g(x)), \quad \text{and} \quad Y \overset{\mathrm{d}}{=} g(X),}$$

where evaluations of function $g(\cdot)$ do not require expensive inversions, $F_Y^{-1}(\cdot)$, as in (1).

- Once we determine the mapping function $g(\cdot)$, then the CDFs $F_X(x)$ and $F_Y(g(x))$ are equal not only in distribution sense but also element-wise.
- It is crucial that function $g(\cdot)$ is as simple as possible.
- What can we say about function $g(\cdot)$?

The Collocation method

- The SC method is used here to efficiently approximate $g(\cdot)$.
- To find a proper mapping function we need to extract some information, via a few inversions $F_Y^{-1}(\cdot)$, from the random variable Y.
- The SC method approximates Y as a function g of X in terms of an expansion in Lagrange polynomials $\ell_i(x_n)$, i.e.,

$$
\begin{aligned}
Y \approx g_N(X) &= \sum_{i=1}^{N} \boxed{F_Y^{-1}(F_X(x_i))}\, \ell_i(X) \\
&= \sum_{i=1}^{N} y_i \ell_i(X), \quad \ell_i(X) = \prod_{j=1, i\neq j}^{N} \frac{X - x_j}{x_i - x_j},
\end{aligned}
$$

- x_i and x_j are so-called *collocation points*.

The Mapping Scheme

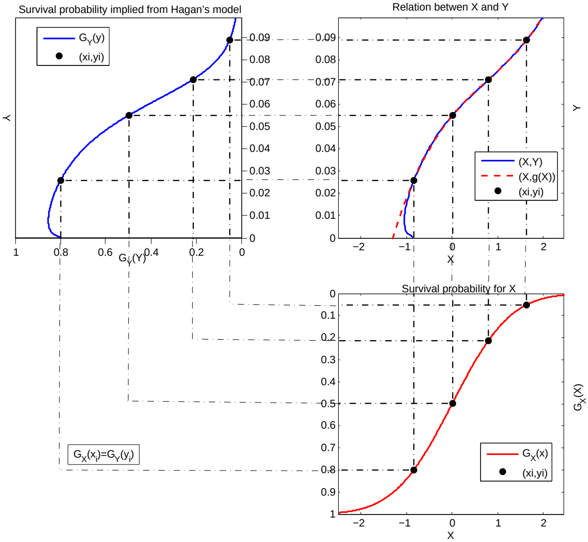

Figure: Illustration of the mappings of Y on $X \sim \mathcal{N}(0,1)$ with a polynomial $g_N(X)$.

Optimal Collocation Points

- The collocation points x_i may be chosen manually (typically unstable polynomial)
- The collocation points can also be determined in an optimal manner based on the orthogonal polynomials.

Theorem (Recurrence in orthogonal polynomials)

For any given density function $f_X(\cdot)$, a unique sequence of monic orthogonal polynomials $p_n(X)$ exists, with $deg(p_n(X)) = n$, which can be constructed as follows,

$$p_{i+1}(X) = (X - \alpha_i)p_i(X) - \beta_i p_{i-1}, \quad i = 0, \dots, N-1, \tag{2}$$

where $p_{-1}(X) \equiv 0$, $p_0(X) \equiv 1$ and where α_i and β_i are the recurrence coefficients,

$$\alpha_i = \frac{\mathbb{E}[Xp_i^2(X)]}{\mathbb{E}[p_i^2(X)]}, \text{ for } i = 0, \dots, N-1, \quad \beta_i = \frac{\mathbb{E}[p_i^2(X)]}{\mathbb{E}[p_{i-1}^2(X)]}, \tag{3}$$

with $\beta_0 = 0$, for $i = 1, \dots, N-1$.

Finding α and β

- Parameters α_i and β_i are completely determined in terms of the moments of random variable X.
- Let us consider the monomials $m_i(X) = X^i$, and define $\mu_{i,j}$ as follows,

$$\mu_{i,j} = \mathbb{E}\left[m_i(X)m_j(X)\right] = \mathbb{E}[X^{i+j}], \quad i,j = 0,\ldots,N. \qquad (4)$$

  From all moments $\mu_{i,j}$ we construct the so-called Gram matrix $M = \{\mu_{i,j}\}_{i,j=0}^{N}$.
- As M is positive definite, we decompose $M = R^{\mathrm{T}}R$.
- As given in Golub and Welsh in 1969 [Golub and Welsch, 1969] and is given by,

$$\begin{aligned} \alpha_j &= \frac{r_{j,j+1}}{r_{j,j}} - \frac{r_{j-1,j}}{r_{j-1,j-1}}, \quad j = 1,\ldots,N, \\ \beta_j &= \left(\frac{r_{j+1,j+1}}{r_{j,j}}\right)^2, \quad j = 1,\ldots,N-1, \end{aligned}$$

  with $r_{0,0} = 1$ and $r_{0,1} = 0$ and where $r_{i,j}$ is the (i,j)-th element of matrix R.

Optimal Collocation points x_i

Theorem (Eigenvalue method)

The zeros x_i, $i = 1, \ldots, N$, of the orthogonal polynomial $p_N(X)$ are the eigenvalues of the symmetric tridiagonal matrix,

$$\widehat{J} := \begin{pmatrix} \alpha_1 & \sqrt{\beta_1} & 0 & 0 & 0 \\ \sqrt{\beta_1} & \alpha_2 & \sqrt{\beta_2} & 0 & 0 \\ 0 & \sqrt{\beta_2} & \alpha_3 & \sqrt{\beta_3} & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \sqrt{\beta_{N-2}} & \alpha_{N-1} & \sqrt{\beta_{N-1}} \\ 0 & 0 & 0 & \sqrt{\beta_{N-1}} & \alpha_N \end{pmatrix},$$

i.e., $\mathbf{x} = eig(\widehat{J})$, with $\mathbf{x} = (x_1, x_1, \ldots, x_N)^{\mathrm{T}}$, α_i and β_i being the coefficients of the three-term recurrence relation (2).

- Once optimal collocation points x_i are known we can simply evaluate polynomial:

$$Y \approx g_N(X) = \sum_{i=1}^{N} F_Y^{-1}(F_X(x_i))\ell_i(X).$$

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The Collocation method

- For a basis of monomials $\mathbf{m}(x) = (1, x, x^2, \ldots, x^{N-1})^{\mathrm{T}}$, the function g can be decomposed as,

$$g_N(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1}, \quad \text{with} \quad g_N(x_i) = y_i, \quad (5)$$

- To find $a_0, a_1, \ldots, a_{N-1}$ we need to solve the following linear system, $V\boldsymbol{a} = \boldsymbol{y}$, i.e.,

$$\begin{pmatrix} 1 & x_1^1 & x_1^2 & \ldots & x_1^{N-1} \\ 1 & x_2^1 & x_2^2 & \ldots & x_2^{N-1} \\ 1 & x_3^1 & x_3^2 & \ldots & x_3^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N-1}^1 & x_{N-1}^2 & \ldots & x_{N-1}^{N-1} \\ 1 & x_N^1 & x_N^2 & \ldots & x_N^{N-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix},$$

- Number of equations in the system will correspond to the number of the collocation points.

The Collocation method- optimal collocation points

- Optimal collocation points x_i need to be determined based on the **moments of** X.
- Once we take X to be normally distributed the collocation points, x_i, are equal to the quadrature points (available in every library).
- In order to use the collocation we only need to calculate a few inversions points

$$y_i = F_Y^{-1}(F_X(x_i)),$$

where points x_i are precalculated.
- Grid-stretching technique allows us to specify a range of the probabilities to which the collocation method is applied. By defining the following mapping:

$$F_Y(Y) = F_X(a + bX) \quad g(X) = F_Y^{-1}(F_X(a + bX)),$$

with $X \sim \mathcal{N}(0, 1)$ we can choose a and b such that an interval $[g_{min}, g_{max}]$ used for mappings will be used. **Needed for handling volatility parameterizations model.**

The Collocation method- Example

- We consider an "expensive" variable $Y \sim \Gamma(5,2)$ and approximate if by a polynomial of standard normals:

$$g_3(X) = \sum_{i=1}^{3} F^{-1}_{\Gamma(5,2)}(F_X(x_i))\ell(X) = a + bX + cX^2.$$

- In the experiment we have generated $M = 10^6$ samples and the corresponding CDF is depicted below.

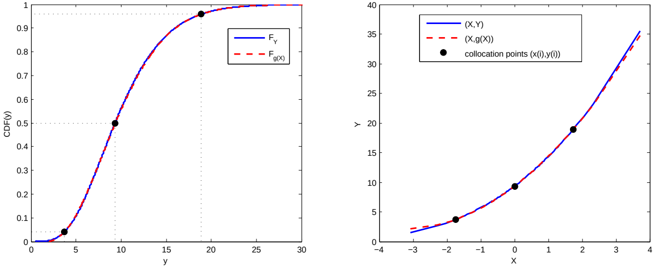

Figure: Left-hand side: Exact CDF for $Y \sim \Gamma(5,2)$ and approx. $g_N(X)$ with $N = 3$ collocation points.

The Collocation method- Example

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$$g_N(X) = \sum_{i=1}^{N} F^{-1}_{\Gamma(5,2)}(F_X(x_i))\ell(X) = a+bX+cX^2+dX^3+eX^4.$$

- In the experiment we have generated $M = 10^6$ samples and the corresponding CDF is depicted below.

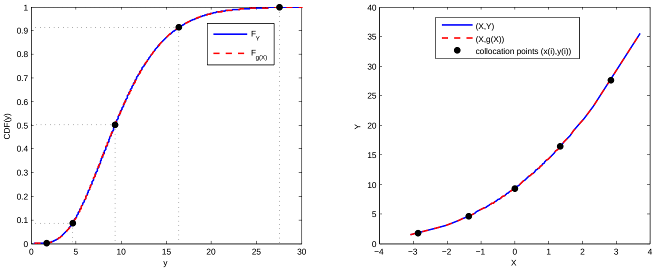


Figure: Left-hand side: Exact CDF for $\Gamma(5,2)$ and approx. with $N = 5$ collocation points.

Arbitrage in the Hagan's model (SABR)

- Its the industry to parameterize the implied volatilities with either the SVI model or Hagan's formula.
- In either parameterization the arbitrage is present.

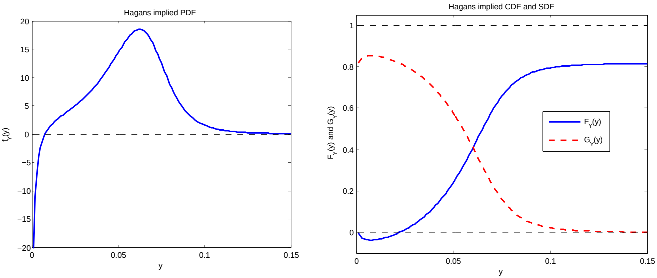


Figure: $\beta = 0.5$, $\alpha = 0.05$, $\rho = -0.7$, $\gamma = 0.4$, $F(t_0) = 0.05$ and $T = 7$. Left: probability density, with deterioration near zero; right: corresponding CDF and SDF (survival distribution function).

- Note that in the Hagan's model deteriorates around an atom at zero.

Hagan's density and Collocation Method

- Since $F_Y(\cdot)$ is not well-defined the inversion $F_Y^{-1}(F_X(x_i))$ will yield incorrect mapping points.
- The survival probability is well-defined and it can be calculated as follows:

$$G_Y(y) = \int_y^{+\infty} f_Y(x)\mathrm{d}x = \boxed{-\frac{\partial V_{\text{call}}(t_0, K)}{\partial K}\bigg|_{K=y}}, \qquad (6)$$

- European-style payoffs can be calculated extremely efficient as:

$$V(t_0, y_0) = \int_0^{\infty} V(T, y) f_Y(y)\mathrm{d}y \approx \int_{G_X^{-1}(1)}^{G_X^{-1}(0)} V(T, g_N(x)) f_X(x)\mathrm{d}x.$$

- When X is a Gaussian variable European put and call option prices are known analytically.

Hagan's density and Collocation Method

- By using the collocation method we want to approximate Hagan's variable by some polynomial of normals:

$$Y \sim a_0 + a_1 X + a_2 X^2 + \dots$$

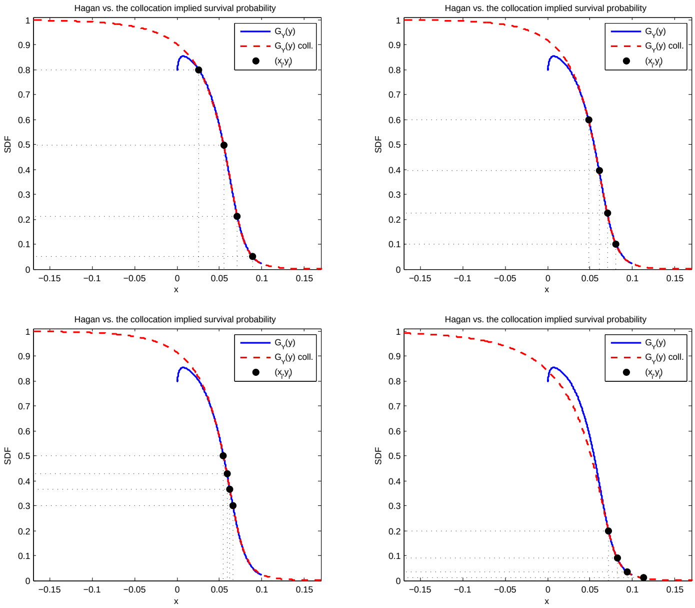
- Under the Hagan's model the collocation mapping is given by:

$$X \approx g_N(X) = \sum_{i=1}^{N} G_Y^{-1}(G_X(x_i))\ell_i(X).$$

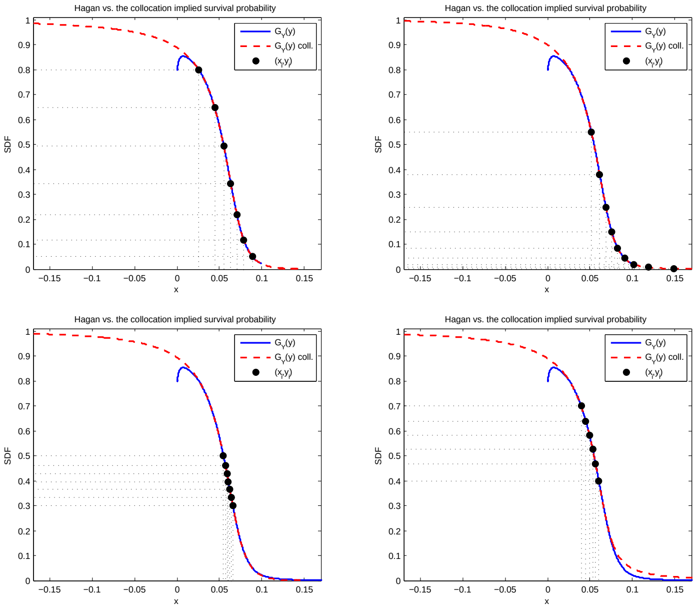
- By using the grid-stretching technique we can specify the range $[g_{\min}, g_{\max}]$ from which Hagan's model performs as expected:

$$y_i = G_Y^{-1}\left(G_{\mathcal{N}(0,1)}\left(a + bx_i\right)\right), \tag{7}$$

Grid-Stretching allows for choosing the mapping region



Grid-Stretching allows for choosing the mapping region



Absorption at zero

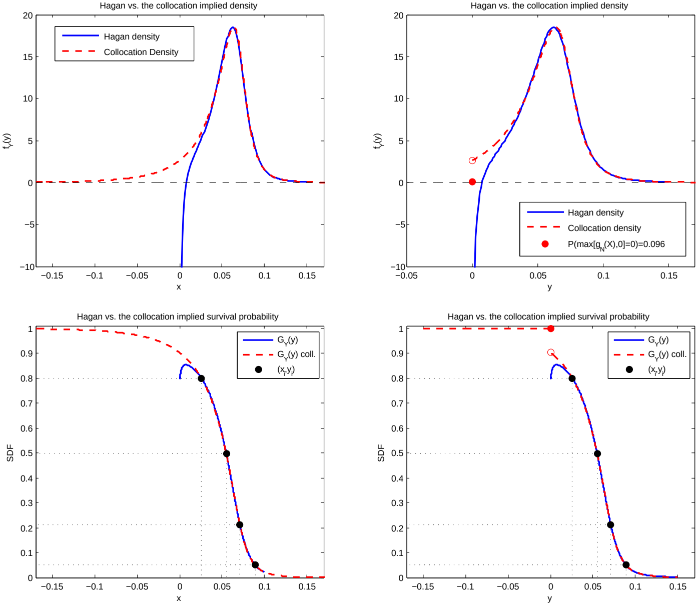
- Note that the projection on a polynomials of normals allows for negative realizations, i.e.: $\mathbb{P}[Y < 0] > 0$.
- The *absorption* at zero feature can be easily incorporated into the methodology by the following enforcement on the function $g_N(X)$:

$$\widehat{g}_N(X) = \begin{cases} g_N(X), & \text{for} \quad g_N(X) > 0 \Leftrightarrow X > g_N^{-1}(0), \\ 0, & \text{for} \quad g_N(X) \leq 0 \Leftrightarrow X \leq g_N^{-1}(0), \end{cases} \tag{8}$$

with $X > g_N^{-1}(0)$ corresponding to the condition of $Y > 0$, with $Y \approx g_N(X)$.
- It is easy to notice that $\widehat{g}_N(X)$ has an atom at $X = g_N^{-1}(0)$ which corresponds to an atom at $Y = 0$. The probability mass is given by:

$$\mathbb{P}[Y = 0] \approx \mathbb{P}[\widehat{g}_N(X) = 0] = \mathbb{P}[g_N(X) \leq 0].$$

Absorption at zero



Pricing of European options

- European option prices are analytical:

Lemma (European call option prices)

With the collocation random variable $X \sim \mathcal{N}(0,1)$ for $g_N(X)$, European call prices are analytically available, and given by:

$$V_{call}(t_0, K) = G_{\mathcal{N}(0,1)}(c_K)\left[\sum_{i=0}^{N-1} a_i \mathbb{E}[X^i | X > c_K] - K\right],$$

with $c_K = g_N^{-1}(K)$, $G_{\mathcal{N}(0,1)}(c_K) = 1 - F_{\mathcal{N}(0,1)}(c_K)$, $\mathbb{E}[X^i|X > c_k]$ the moments of the truncated normal.

- Expectations $\mathbb{E}[X^i|X > c_K]$ are also known analytically.
- Coefficients a_i and the function $g_N(x)$ are known using the collocation method.
- Option pricing is within a split-second.

The mapping algorithm

- Take $Y(\alpha, \beta, \rho)$ to be SABR/Hagan and X to be normally distributed r.v.
- Choose N collocation points and determine optimal collocation points $[x_1, x_2, \ldots, x_N]$ (quadrature points).
- Specify g_{max} and g_{min} (the range in which we “like” the performance of Hagan)
- Determine a, b and find $y_i = G_Y^{-1}\left(G_X\left(a + bx_i\right)\right)$
- For given x_i and y_i find $a_0, \ldots, a_{N-1}$ and construct polynomial

$$Y \approx a_0 + a_1 X + a_2 X^2 + \ldots$$

- Calculate analytically option prices and implied volatilities.

Model Recalibration

- To enhance the results we define an optimization procedure in which we determine the set, $\widehat{\Omega} = [\widehat{\beta}, \widehat{\alpha}, \widehat{\rho}, \widehat{\gamma}]$, of parameters so that the volatilities from the market and by the collocation method agree, i.e.

$$\min_{\widehat{\Omega}} \sum_{\overline{K}} \left(\sigma_{Mrkt}(\overline{K}) - \sigma_{g(X,\widehat{\Omega})}(\overline{K}) \right)^2 . \tag{9}$$

  Typically only a few strikes $\overline{K}$ are used in this calibration procedure.

- We can also enforce the martingale property via:

$$\mathbb{E}[g_N(X)] = \sum_{i=0}^{N-1} a_i \mathbb{E}[X^i] = S_0. \tag{10}$$

Numerical Experiments

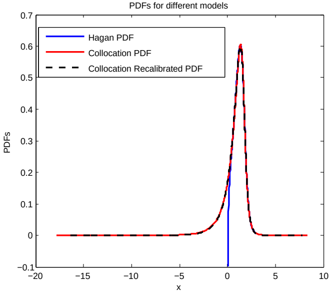
- We consider the following three sets of parameters

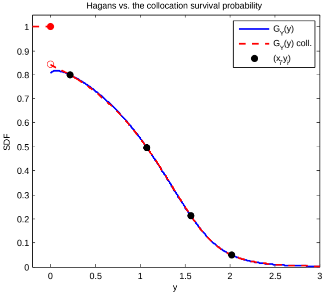
Parameters:	β	α (ATM)	ρ (Corr)	γ (vol-vol)	$F(t_0)$	T
Set I as in [1]	0.6	0.25	-0.8	0.3	1	10
Set II as in [2]	0.25	0.35	-0.1	1	1	1
Set III as in [3]	0.2	0.26	-0.5	0.35	1	15

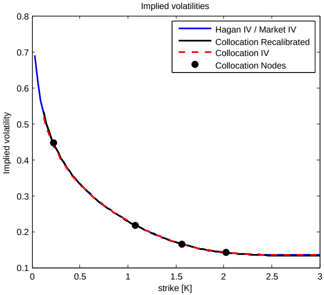
Table: Model parameters chosen in the experiments. Antonov [Antonov and Spector,], Hagan [Hagan et al., 2014] and Balland [Balland and Tran,]

- We consider simple projection and the re-calibrated variant.

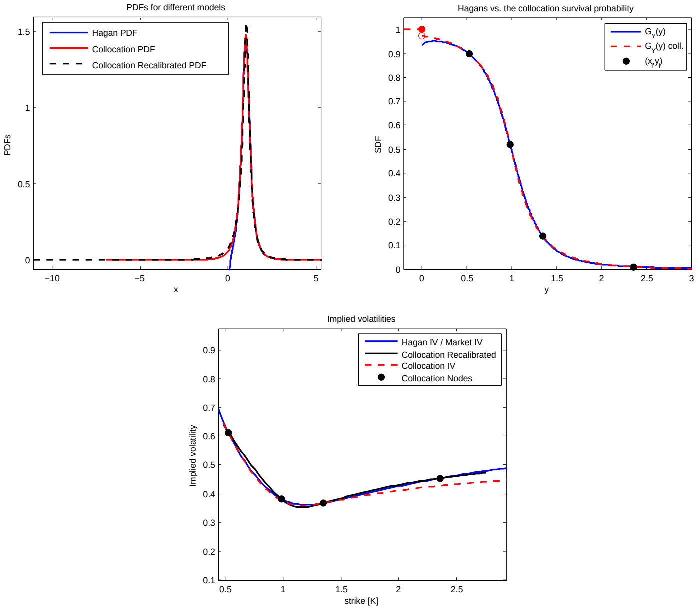
Numerical Experiments: Antonov



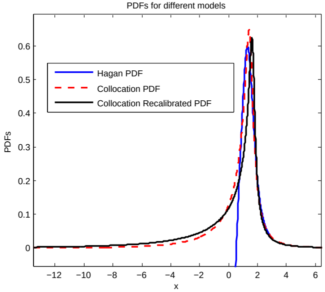


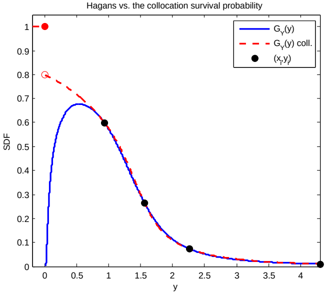


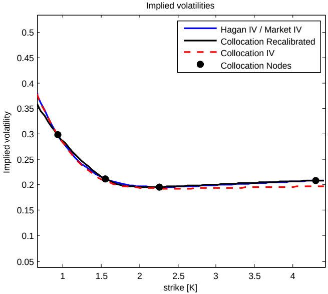
Numerical Experiments: Hagan



Numerical Experiments: Balland







The Key Points

- We have discussed an application of the stochastic collocation method for obtaining an arbitrage-free density based on Hagan's formula.
- The method relies on the availability of a survival distribution function, not necessarily well-defined on the whole domain, which is projected on a Gaussian variable.
- The technique presented gives implied volatilities in accordance with those obtained by the model, however, in some cases a re-calibration step is required to enhance the fit.
- The method is easy to implement as it only relies on Lagrange interpolation and solving a linear system of equations.
- From computational perspective the model is extremely cheap to evaluate as it only involves a few inversions of the CDF/SDF of the original distribution.

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