Analytical Credit VaR

a simple analytical approach to credit portfolio modeling

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Background

Let us consider a portfolio of credit risky facilities with loss functions $\{l_i\}(\epsilon_i)$ at horizon (one year) being a function of random variables (normalized asset returns) $\{\epsilon_i\}$. Dependencies within the portfolio are modeled by means of a set of common factors $\{\eta_k\}$:

$$\epsilon_i = \rho_i \sum_k (\beta_i)_k \eta_k + \sqrt{1 - \rho_i^2} \xi_i \beta \quad (1)$$

The economic capital of the portfolio is defined as na α - quantile (usually set to 99.9% or higher) of the portfolio loss distribution $L = \sum_i l_i$ relative to the expected loss of the portfolio:

$$\mathrm{EC} = q_\alpha[L] - \mathrm{E}[L] \quad (2)$$

Background

Capital allocation using Euler principle (*Tasche, 2008*):

$$\mathrm{EC} = \sum_i \mathrm{ec}_i, \qquad \mathrm{ec}_i = w_i \frac{\partial}{\partial w_i} \mathrm{EC} \tag{3}$$

Solution of (2) can be written as:

$$q_\alpha[L] = q_\alpha[L_{1f}] + \delta q_\alpha[\delta L_{mf}], \qquad L_{1f} = \mathrm{E}[L|\eta_{1f}], \quad \mathrm{E}[L_{mf}] = 0 \tag{4}$$

The single factor is constructed as a linear combination of the systematic factors $\eta_{1f} = \sum_k \alpha_k \eta_k$ with the normalization condition $\sum_k \alpha_k^2 = 1$.

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Background

The conditional expectation series expansion technique (*Voropaev, 2011*) can be applied to facilitate calculations of $L_{1f}(\eta_{1f})$:

$$\bar{l}_i(\eta_{1f}) = \sum_{n=0}^{\infty} \frac{(\rho_i \vec{\beta}_i \vec{\alpha})^n}{n!} l_i^{(n)} \mathrm{He}_n(\eta_{1f}), \tag{5}$$

where

$$l_i^{(n)} = \int l_i(\epsilon) \mathrm{He}_n(\epsilon) \frac{e^{-\epsilon^2/2}}{\sqrt{2\pi}} \mathrm{d}\epsilon \tag{6}$$

Systematic vs. idiosyncratic risk

Treating the systematic and the idiosyncratic risk equally not only simplifies the model structure, but also allows straightforward incorporation of the borrower concentration effects into the portfolio risk metrics. Let us introduce:

$$\vec{\rho}_i = (\rho_i\beta_{i1}, \rho_i\beta_{i2}, \ldots, \rho_i\beta_{iM}, 0, \ldots, \sqrt{1-\rho_i^2}, \ldots, 0) \quad (7)$$

$$\vec{\alpha} = (\alpha_1, \ldots, \alpha_M, \alpha_{M+1}, \ldots, \alpha_{M+N}) \quad (8)$$

$$r_i = \vec{\rho}_i \cdot \vec{\alpha} = \sum_{k=1}^{M+N} \rho_{ik}\alpha_k \quad (9)$$

The single factor approximation L_{1f} of the portfolio loss:

$$L_{1f} = \sum_i \sum_{n=0}^{\infty} \frac{r_i^n}{n!} l_i^{(n)} \mathrm{He}_n(\eta_{1f}) \quad (10)$$

Single factor approximation

Assuming that the multi-factor corrections give positive contribution to the α-quantile of the portfolio loss distribution L:

$$q_\alpha[L] \approx \max_{1f} q_\alpha[L_{1f}] \tag{11}$$

The optimization problem:

$$\nabla_{\vec{\alpha}} \sum_i \bar{l}_i(r_i) = 0, \qquad r_i = \vec{\alpha} \cdot \vec{\rho}_i \tag{12}$$

Can be solved numerically starting with

$$\vec{\alpha}_0 = \frac{\vec{p}_0}{\|\vec{p}_0\|}, \qquad \vec{p}_0 = \sum_i \frac{\partial \bar{l}_i(r_i)}{\partial r_i} \vec{\rho}_i \Big|_{r_i=0} \tag{13}$$

Single factor approximation

The optimal single factor defined by (12) leads to another simplification for the portfolio capital allocation problem (3). The individual capital contributions

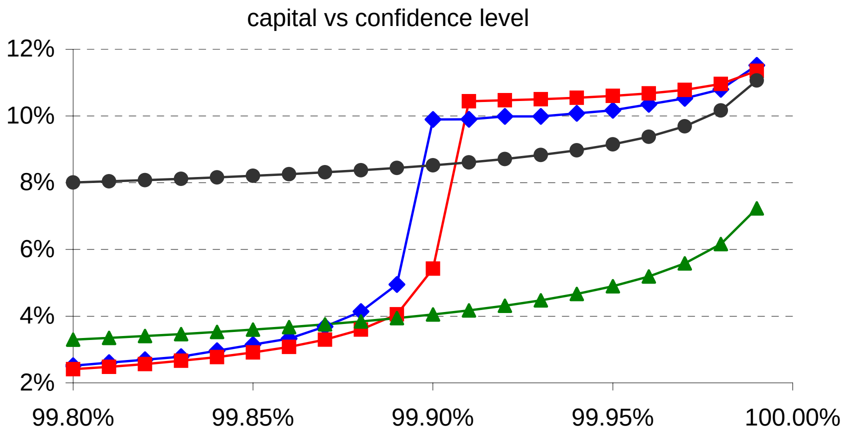
$$\mathrm{ec}_i = w_i \frac{\partial}{\partial w_i} \mathrm{EC} = w_i \frac{\partial}{\partial w_i} \sum_i \left(\bar{l}_i(r_i) - \mathrm{E}[l_i] \right) \tag{14}$$

can be written as

$$\mathrm{ec}_i = \bar{l}_i - \mathrm{E}[l_i] + \sum_j \frac{\partial \bar{l}_i}{\partial r_i} \vec{\rho}_j \cdot w_i \frac{\partial}{\partial w_i} \frac{\vec{p}}{\|\vec{p}\|} = \bar{l}_i(\vec{\alpha}\vec{\rho}_i) - \mathrm{E}[l_i] \tag{15}$$

where the third term can be shown to be zero.

Benchmarking



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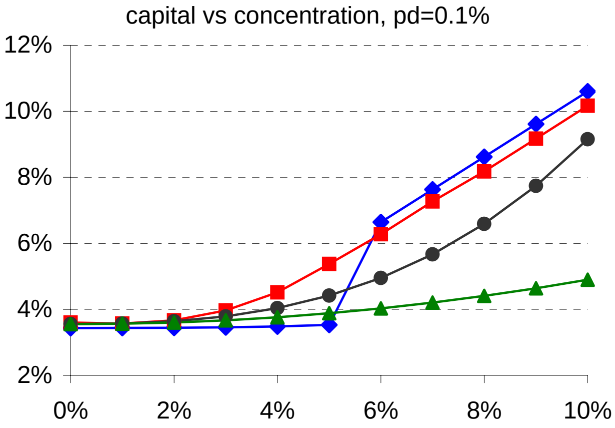

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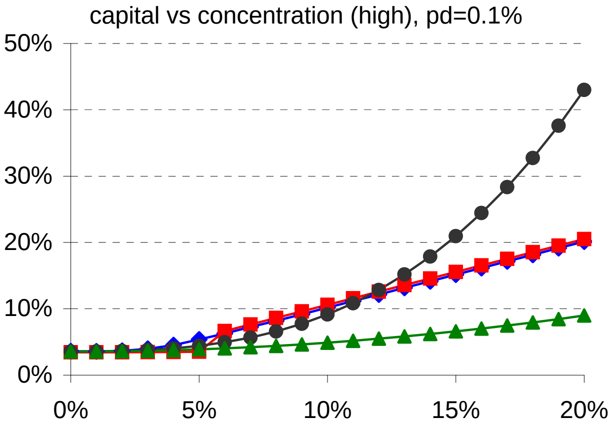

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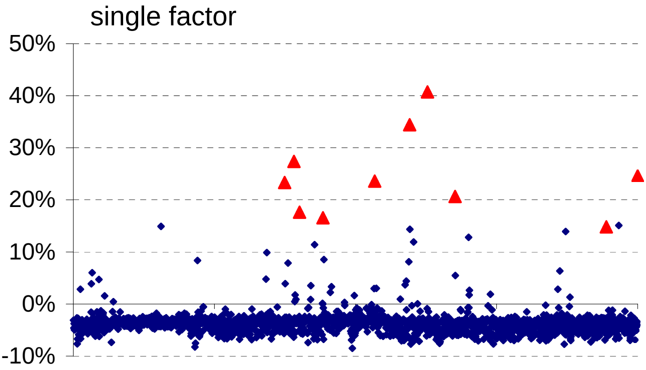
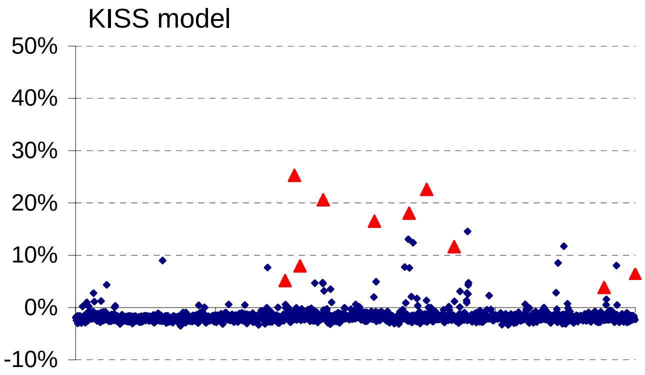


Benchmarking



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Q&A