Analytical Credit VaR
a simple analytical approach to credit portfolio modeling

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Model Validation
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Let us consider a portfolio of credit risky facilities with loss functions \( \{l_i\}(\epsilon_i) \) at horizon (one year) being a function of random variables (normalized asset returns) \( \{\epsilon_i\} \). Dependencies within the portfolio are modeled by means of a set of common factors \( \{\eta_k\} \):

\[
\epsilon_i = \rho_i \sum_k (\beta_i)_k \eta_k + \sqrt{1 - \rho_i^2} \xi_i \beta
\]  

(1)

The economic capital of the portfolio is defined as the \( \alpha \) - quantile (usually set to 99.9% or higher) of the portfolio loss distribution \( L = \sum_i l_i \) relative to the expected loss of the portfolio:

\[
EC = q_\alpha[L] - E[L]
\]  

(2)
Capital allocation using Euler principle (Tasche, 2008):

\[ EC = \sum_i ec_i, \quad ec_i = w_i \frac{\partial}{\partial w_i} EC \] (3)

Solution of (2) can be written as:

\[ q_\alpha[L] = q_\alpha[L_{1f}] + \delta q_\alpha[\delta L_{mf}], \quad L_{1f} = E[L|\eta_{1f}], \quad E[L_{mf}] = 0 \] (4)

The single factor is constructed as a linear combination of the systematic factors \( \eta_{1f} = \sum_k \alpha_k \eta_k \) with the normalization condition \( \sum_k \alpha_k^2 = 1 \).
Background

The conditional expectation series expansion technique (Voropaev, 2011) can be applied to facilitate calculations of \( L_{1f}(\eta_{1f}) \):

\[
\bar{l}_i(\eta_{1f}) = \sum_{n=0}^{\infty} \frac{(\rho_i \beta_i \bar{\alpha})^n}{n!} l_i^{(n)} \ln(\eta_{1f}), \quad (5)
\]

where

\[
l_i^{(n)} = \int l_i(\epsilon) \ln(\epsilon) \frac{e^{-\epsilon^2/2}}{\sqrt{2\pi}} d\epsilon \quad (6)
\]
Systematic vs. idiosyncratic risk

Treating the systematic and the idiosyncratic risk equally not only simplifies the model structure, but also allows straightforward incorporation of the borrower concentration effects into the portfolio risk metrics. Let us introduce:

\[ \vec{\rho}_i = (\rho_i \beta_1, \rho_i \beta_2, \ldots, \rho_i \beta_M, 0, \ldots, \sqrt{1 - \rho_i^2}, \ldots, 0) \] (7)

\[ \vec{\alpha} = (\alpha_1, \ldots, \alpha_M, \alpha_{M+1}, \ldots, \alpha_{M+N}) \] (8)

\[ r_i = \vec{\rho}_i \cdot \vec{\alpha} = \sum_{k=1}^{M+N} \rho_{ik} \alpha_k \] (9)

The single factor approximation \( L_{1f} \) of the portfolio loss:

\[ L_{1f} = \sum_i \sum_{n=0}^{\infty} \frac{r_i^n}{n!} l_i^{(n)} H(n, \eta_{1f}) \] (10)
Single factor approximation

Assuming that the multi-factor corrections give positive contribution to the $\alpha$-quantile of the portfolio loss distribution $L$:

$$q_\alpha[L] \approx \max_{1f} q_\alpha[L_{1f}]$$  \hspace{1cm} (11)

The optimization problem:

$$\nabla \tilde{\alpha} \sum_i \tilde{l}_i(r_i) = 0, \quad r_i = \tilde{\alpha} \cdot \tilde{\rho}_i$$  \hspace{1cm} (12)

Can be solved numerically starting with

$$\tilde{\alpha}_0 = \frac{\tilde{p}_0}{\|\tilde{p}_0\|}, \quad \tilde{p}_0 = \sum_i \left. \frac{\partial \tilde{l}_i(r_i)}{\partial r_i} \tilde{\rho}_i \right|_{r_i=0}$$  \hspace{1cm} (13)
The optimal single factor defined by (12) leads to another simplification for the portfolio capital allocation problem (3). The individual capital contributions

\[ ec_i = w_i \frac{\partial}{\partial w_i} EC = w_i \frac{\partial}{\partial w_i} \sum_i (\bar{l}_i(r_i) - E[l_i]) \] (14)

can be written as

\[ ec_i = \bar{l}_i - E[l_i] + \sum_j \frac{\partial \bar{l}_i}{\partial r_i} \rho_j \cdot w_i \frac{\partial}{\partial w_i} \frac{\rho}{\|\rho\|} = \bar{l}_i(\bar{\alpha}\bar{\rho}_i) - E[l_i] \] (15)

where the third term can be shown to be zero.
Benchmarking capital vs confidence level

Monte Carlo simulation
KISS model
2nd order multi-factor
3rd order multi-factor

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Benchmarking

capital vs concentration, pd=0.1%

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Analytical Credit VaR
Benchmarking
capital vs concentration (high), pd=0.1%
Benchmarking

single factor

-10%
0%
10%
20%
30%
40%
50%
Benchmarking

KISS model

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References
