MECHANISMS FOR NO-ARBITRAGE TERM-STRUCTURE MODELLING, WITH APPLICATIONS TO INTEREST-RATES AND REALIZED VARIANCE

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A reversal of ECB policy in the next months,
what consequence for our positions?

A continued rise in volatility during these next weeks,
how to quantify the effects?

PLAN

Develop a framework to handle these questions as follows.

- **Step 1.** Establish **construction methods** for **no-arbitrage** dynamics of **term-structures of forward rates** (forward interest-rates, forward relized-variances, etc).

- **Step 2.** Refine the Step 1 mechanisms to assure compliance of **instantaneous rates** (short rate, instantaneous realized-variance, etc) with empirically observed **stylized facts**:
  - finite time-horizon **mean reversion**.
  - finite time-horizon **positivity** (respectively **boundedness from below**).
  - **non-exploding prices** of corresponding primary instruments (bonds,variance swaps, etc).

- **Step 3.** Assure **tractability** of the constructions, as well as its **transfer** to derivatives valuation and hedging ... in concrete cases, moreover.

Benchmark for tractability: Vasicek model of short-rate.

MODELLING FRAMEWORK: HIGHLY-STYLIZED, I.

Work on a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_u)_{u \in [0,T^*]}, P)$ where $T^* \in [0, \infty]$.

- **Given:** the family of **forward-rate processes**

$$(f_{t,T})_{t \in [0,T]}\,, \quad \text{for every } T \in [0, T^*)\,.$$

- **Consider:** the family of **term-structure processes**, for every $T \in [0, T^*)$ given by

$$(Y_{t,T})_{t \in [0,T]}\,, \quad \text{where } Y_{t,T} = g\big(\int_{[t,T]} f_{t,u}\, du \big)\,,$$

for fixed and sufficiently nice g : (range of the $f_{t,T}$) $\to$ **R**.

- **Step 1. Construction of $(\mathbf{F}, Q)$-no-arbitrage dynamics** by asking the validity of the **expectations hypothesis**

$$Y_{t,T} = E^Q\big[g(R_T - R_t) \,|\, \mathscr{F}_t \big]\,, \;\; t \in [0,T]\,,$$

where

$$R_u = \int_{[0,u]} f_{s,s}\, ds\,, \;\; u \in [0, T^*)\,.$$

for any $T \in [0, T^*]$, on measure change from P to an equivalent Q.

MODELLING FRAMEWORK: HIGHLY-STYLIZED, II.

Continue in the setting of Step 1.

- Focus on the **instantaneous rate**, namely

$$(f_{u,u})_{u\in[0,T^*)}\ ,$$

  and look at its $(\mathbf{F}, P)$-**dynamics**.

- **Step 2.** Effect **compliance with stylized facts** of the Step 1 construction by asking validity of:
  - **Mean reversion (MR).** We have

$$\lim_{u\uparrow T^*} f_{u,u} = f_{T^*,T^*}\ ,$$

    in a cup or L^1 sense w.r.t. P.
  - **Positivity** (P_ε). For fixed $\varepsilon \geq 0$, we have

$$f_{u,u} \geq \min\{0, f_{0,u} - \varepsilon\}\ ,$$

    for every $u \in [t_0, T_0] \subseteq [0, T^*)$, in a P-a.s. sense.

MODELLING FRAMEWORK: HIGHLY-STYLIZED, III.

- **Specialization** of the modelling framework by choice of g:

$$\begin{pmatrix}\text{realized-variance} \\ \text{modelling}\end{pmatrix} \longleftrightarrow \quad g = \mathrm{id}\,.$$

$$\begin{pmatrix}\text{interest-rate} \\ \text{modelling}\end{pmatrix} \longleftrightarrow \quad g = 1/\exp\,.$$

- **For concreteness' and definiteness' sake:** Focus in the rest of the talk on

$$g = 1/\exp\,,$$

the **interest-rate case**.

MODELLING FRAMEWORK, TRANSLATED

Construct short rate r as a process on a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_u)_{u \geq 0}, P)$:

$$r = (r_u)_{u \geq 0},$$

with the **construction to satisfy** 4 conditions:

- **No-arbitrage**: There is an EMM (equivalent martingale measure) $Q \sim P$ such that for all points in time $t \leq T$:

$$P_{t,T} = \begin{pmatrix} \text{time-}t \text{ price of} \\ \text{maturity-}T \\ \text{zerobond} \end{pmatrix} = E^Q\big[\exp\big(-\int_t^T r_u\,du\big) \,\big|\, \mathcal{F}_t\big];$$

  here then $f_{t,u} = -\partial_T \log(P_{t,T})|_{T=u}$ and vice versa.
- **FIT**: The (observed) time-0 prices $(P_{0,T})_{T \geq 0}$ satisfy

$$P_{0,T} = E^Q[\exp(-\int_0^T r_u\,du)], \quad T \geq 0.$$

- **Stylized Fact** (P_ε): Have $r_u \geq \min\{0, f_{0,u} - \varepsilon\}$, for all $u \in [t_0, T_0] \subseteq [0, T^*) \subseteq \mathbf{R}_{\geq 0}$.
- **Stylized Fact (MR)**: Have existence (in ucp or L^1) and (P_ε) of $r_{T^*} = \lim_{u \to T^*} r_u$.

BENCHMARK MODEL, I: NO-ARBITRAGE.

The Vasicek model postulates short rate dynamics of the form

$$r_t = m_t + \int_0^t \sigma_{s,t}\, dW_s\,, \quad t \in \mathbf{R}_{\geq 0}$$

driven by $(\mathbf{F}, P)$-Brownian motion W, with vol structure for fixed a, $\overline{\sigma} \in \mathbf{R}_{>0}$ given by:

$$\sigma_{s,t} = \overline{\sigma} \exp\left(-a(t-s)\right), \quad s \leq t\,.$$

Conditions **No-arbitrage** and **FIT** hold with the choice:

$$\begin{aligned} m_t &= f_{0,t} + \int_0^t \partial_T \Theta_W(-\Sigma_{s,T})\Big|_{T=t}\, ds \\ &= f_{0,t} + \frac{1}{2}(\overline{\sigma}/a)^2(1-e^{-at})^2\,, \end{aligned}$$

for all $t \in \mathbf{R}_{\geq 0}$, where $f_{0,t} := -\partial_t P_{0,t}$.

BENCHMARK MODEL, II: STYLIZED FACTS.

Continue with **No-arbitrage** and **FIT** to hold.

- **Properties I**: Have

$$r_u \sim N(\mu_u, \mathrm{var}_u),$$

where $\mu_u = m_u$ and $\mathrm{var}_u = \overline{\sigma}^2/(2a)(1 - e^{-2au})$.

- **Properties II**: Assuming existence of $f_{0,\infty} = \lim_{t\to\infty} f_{0,t}$, we have existence of $r_\infty = \lim_{u\to\infty} r_u$ with

$$r_\infty \sim N(\mu_\infty, \mathrm{var}_\infty)\,,$$

where $\mu_\infty = \lim_{u\to\infty} \mu_u = f_{0,\infty} + (1/2)(\overline{\sigma}/a)^2$ and $\mathrm{var}_\infty = \lim_{u\to\infty} \mathrm{var}_u = \overline{\sigma}^2/(2a)$.

- **Problem 1**: Positivity of r_u and r_∞ systematically violated!

- **Problem 2**: Exploding bond prices, otherwise!

BENCHMARK MODEL, III: (P_ε) failure.

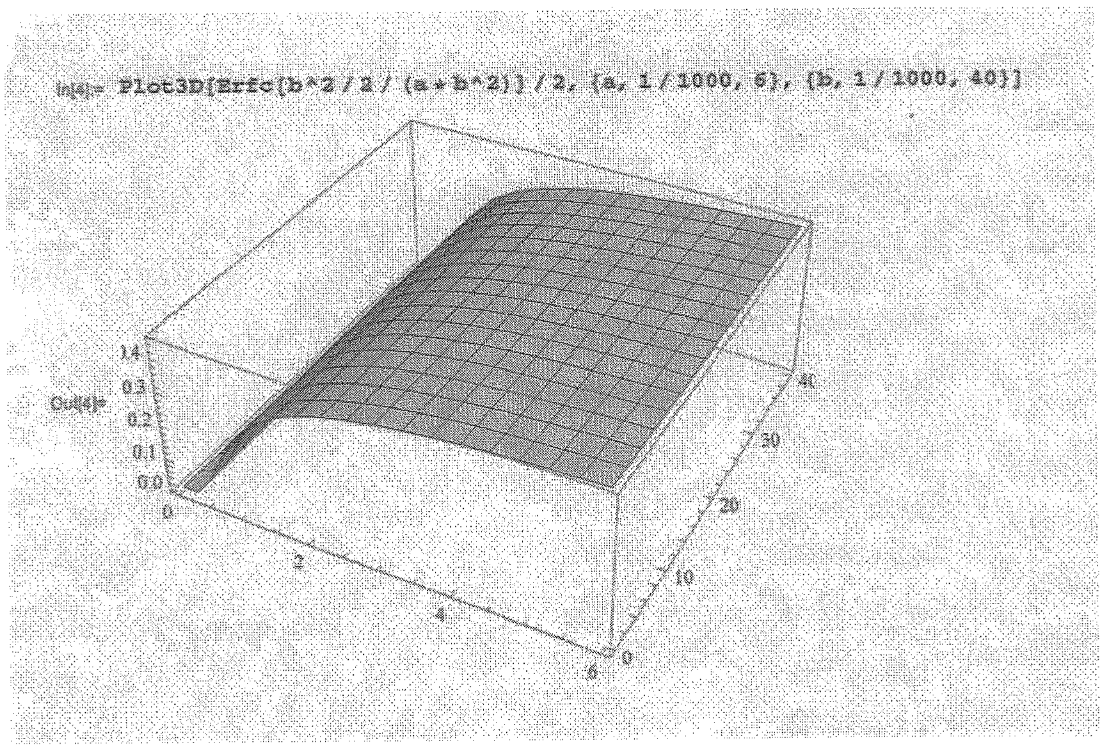


Figure 1. $P(r_\infty < 0)$ in dependency on Vasicek parameters a and $b = \overline{\sigma}/a$.

DRAWING ON WORK OF EBERLEINs

To overcome negativity problems of Vasicek short rates we make use of work of Eberlein et al's as follows.

- **Basic idea**: Keep the overall form of the Vasicek model dynamics for r but exchange as its driver Brownian motion W by a general semimartingale L.
- **More precisely**, consider dynamics

$$r_u = a_u + \int_0^t \alpha_{s,t}\, ds + \int_0^t \sigma_{s,t}\, dL_s\,, \quad t \in \mathbf{R}_{\geq 0}\,,$$

with L a $(\mathbf{F}, P)$-semimartingale and (here as well as in the rest of the talk!) $a : \mathbf{R}_{\geq 0} \to \mathbf{R}$ and α, $\sigma : \mathbf{R}^2_{\geq 0} \to \mathbf{R}$ sufficiently nice (processes or even just) functions.

- **Problem**: Already **No-arbitrage** is **unachievable** in general by structural reasons: L is too general a semimart!
- **Remedy**: Have to restrict the generality of L and resort to **constructing** appropriate semimartingale drivers L.

DRIVING PROCESSES, I.

- **Basic insight**: Already the class of **PII-semimartingales** furnishes appropriate semimartingale drivers L.
- **The construction of PII-semimartingales** L is in terms of triples (b, c, ν), to be called *PII-triples*, where:
  - $b \in L^1_{\mathrm{loc}}(\mathbf{R}_{\geq 0})$,
  - $c \in L^1_{\mathrm{loc}}(\mathbf{R}_{\geq 0})$ and $c \geq 0$,
  - $\nu = \{\nu_s(dx) \times ds\}_{s>0}$ is a (predictable) random measure on $\mathbf{R} \times \mathbf{R}_{>0}$ with all $(|x|^2 \wedge 1)\nu_s(x)$ integrable on $\mathbf{R}$,

  and is (morally) effected by associating with (b, c, ν) the process given by:

$$L_t = \int_0^t b^*_s \, ds + L^{\mathrm{c}}_t + L^{\mathrm{pd}}_t \,, \quad t \in \mathbf{R}_{\geq 0} \,,$$

  where $b^*_s = b_s + (x - \mathbf{1}_{\{|x| \leq 1\}}) * \nu_s$ and where

$$L^{\mathrm{c}}_t = \int_0^t \sqrt{c_s} \, dW_s \,,$$

$$L^{\mathrm{pd}}_t = \int_0^t \int_{\mathbf{R}} x(\mu - \nu)(dx, ds) \,,$$

  for any $t \in \mathbf{R}_{\geq 0}$, are the *continuous martingale* (*Gaussian*) *part* of L and the *purely-discontinuous martingale part* of L respectively (here μ is the jump measure).

DRIVING PROCESSES, II.

- The **driving processes** L to be chosen encompass Brownian motion with drift.
- Starting from an arbitrary **PII-triple** (b, c, ν), where:
  - $b = (b_s)_{s \geq 0} \in L^1_{\mathrm{loc}}(\mathbf{R}_{\geq 0})$,
  - $c = (c_s)_{s \geq 0} \in L^1_{\mathrm{loc}}(\mathbf{R}_{\geq 0})$ and $c \geq 0$,
  - $\nu = \{\nu_s(dx) \times ds\}_{s>0}$ is a (predictable) random measure on $\mathbf{R} \times \mathbf{R}_{>0}$ with all $(|x|^2 \wedge 1)\nu_s(x)$ integrable on $\mathbf{R}$,
- they are characterized by the **Fourier transforms of their laws**:

$$E[\exp(zL_u)] = \exp\left(\Theta_{L,u}(z)\right),$$

where

$$\Theta_{L,u}(z) = \int_{[0,u]} \vartheta_{L,s}(z)\, ds\,,$$

with

$$\vartheta_{L,s}(z) = zb_s + \tfrac{1}{2}z^2 c_s + \int_{\mathbf{R}} \left(e^{zx} - 1 - z\mathbf{1}_{\{|x| \leq 1\}}\right) \nu(dx, ds)\,,$$

for complex z in some open neighborhood of $\sqrt{-1}\,\mathbf{R}$.

PEDAGOGICAL DEVICE re (EMM)

We consider –initially– the situation:

$$Q = P \, ,$$

where the (given) **statistical measure** P is taken to also furnish an **EMM**.

Will later indicate how to get beyond this convenient but particular arrangement!

STEP 1: THE SHORT-RATE MECHANISM, I.

- **Idea**: Assure dynamics with **No-arbitrage** by adaption of drift.
- **Theorem 1 (Short rate mechanism):** *There is a construction to associate with (sufficiently nice) pairs* $(\sigma,\ (b,c,\nu))$ *of*
  - $\sigma : \mathbf{R}^2_{\geq 0} \to \mathbf{R}_{\geq 0}$ *volatility structure,*
  - (b,c,ν) *PII-triple*

  a dynamics of the short rate r *satisfying* **No-arbitrage** *and* **FIT**. *Explicitly the construction proceeds by* **adaption of the drift** *by way of the definition*

$$r_t = m_t + \int_0^t \sigma_{s,t}\, dL_s\,,\quad t \in \mathbf{R}_{\geq 0}\,,$$

where $L = L(b,c,\nu)$ *and*

$$m_t = f_{0,t} + \int_0^t \partial_T \Theta_{L,s}(-\Sigma_{s,T})\big|_{T=t}\, ds$$

setting $\Sigma_{s,T} = \int_s^T \sigma_{s,u}\, du$ *and with*

$$\Theta_{L,s}(z) = z\int_0^s b_u\, du + \frac{z^2}{2}\int_0^s c_u\, du + \int_0^s \int_{\mathbf{R}} (e^{zx} - 1 - z\mathbf{1}_{\{|x|\leq 1\}})\nu(dx,ds)\,,$$

for complex z *in a suitable open neighborhood of* $\sqrt{-1}\,\mathbf{R}$.

STEP 1: THE SHORT-RATE MECHANISM, II.

- The **key step** for establishing the S-R M is to assure validity of the **expectations hypothesis**, namely

$$P_{t,T} = E^{Q}\big[\exp\big(-\int_t^T r_u\,du\big)\,\big|\,\mathcal{F}_t\big]\,,\quad t \in [0,T]\,,$$

  for any $T < T^*$.

- The **key tool** for this is furnished by the following result.

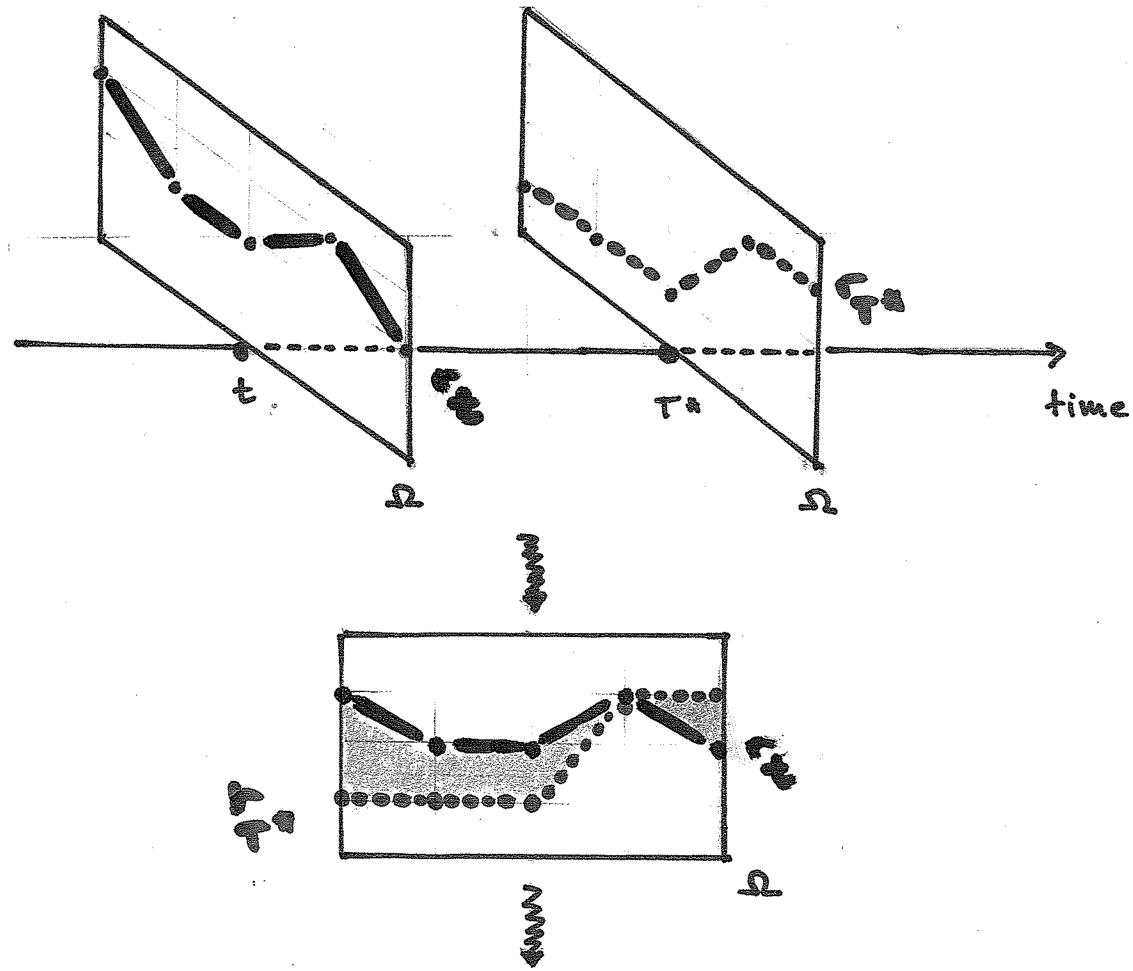
  Lemma: *For PII-semimarts L we have*

$$E\big[\exp\big(\int_t^T f(s)\,dL_s\big)\,\big|\,\mathcal{F}_t\big] = \exp\big(\int_t^T \vartheta_{L,s}\big(f(s)\big)\,ds\big)\,,$$

  for any $t \in [0,T]$ and any càg map $f : [t,T] \to \mathrm{Dom}(\Theta_{L,T})$.

- **Credits:** The S-R M provides a perspective on the vintage 2005 U Freiburg PhD thesis of W. Kluge, written under the direction of E. Eberlein; this perspective was developed in the 2009 MSc-thesis of L. Slamova.

STEP 2: ADD MEAN-REVERSION, I.



$$E[\|r_{T^*} - r_t\|] \overset{!}{\lesssim} \varepsilon$$

as $t \nearrow T^*$.

STEP 2: ADD MEAN-REVERSION, II.

- Address **finite-horizon mean-reversion** of $(r_u)_{u \geq 0}$ by asking, for fixed $T^* \in (0, \infty]$ and $\varepsilon \geq 0$, the following condition:

  (MR) $E[|r^*_T - r_t|] < \varepsilon$, for every $t \in [T^* - \delta, T^*]$, for some $\delta = \delta(r, T^*, \varepsilon) > 0$.

- **Task:** Incorporate (MR) in the Step 1 no-arbitrage correspondence $(\sigma, L) \mapsto r(\sigma, L)$ of Theorem 1.

- **Theorem 2:** *If L has first (and second) order moments w.r.t. the chosen EMM Q, then $r(\sigma, L)$ also satisfies (MR) under 4 additional conditions centered on σ being contained in $L^1(0, \infty)$ and $L^2(0, \infty)$ with 'good' respective Cauchy-condition-type convergence properties.*

STEP 2: ADD MEAN-REVERSION, III.

Range of applicability of the **Theorem 2** incorporation of (MR) in the Step 1 no-arbitrage correspondence.

- **Processes:**
  - **Does not apply** to α-stable processes L, but
  - **does apply** to every process L whose moment generating function $z \mapsto E[\exp(zL_1)]$ is finite on an open neighborhood of $\sqrt{-1}\,\mathbf{R}$, as, e.g., for $L \in \{\mathrm{GIG}, \mathrm{NIG}\}$.

- **Vol-structures:**
  - **Does not apply** to $\sigma = \mathrm{const} > 0$, but
  - **does apply** to the vol-structures given by

$$\sigma_{s,u} = \overline{\sigma} \exp\left(-a(u-s)\right), \quad (1)$$

$$\sigma_{s,u} = \overline{\sigma} \exp\left(-a(u-s)\right) \frac{1+\gamma u}{1+\gamma s}, \quad (2)$$

for any $s \leq u$, for arbitrary fixed real $\overline{\sigma}$, $a > 0$ and γ.

STEP 2: ADD POSITIVITY, I.

Starting from arbitrary but sufficiently regular pairs

$$(\sigma, L) \text{ where } \begin{array}{l} \sigma \text{ vol-structure on } [0, T^*]^2 \\ L \text{ PII-semimartingale} \end{array}$$

we have constructed

- a **no-arbitrage correspondence** of $(f_{s,u})_{s \in [0,u]}$, the forward-rate processes,
- which induces a **no-arbitrage short-rate process** $r = r(\sigma, L)$
- that satisfies finite-horizon **mean-reversion (MR)**, under integrability conditions on σ, L.

We **wish to incorporate** in these correspondences in addition

- compliance with **rising-rate scenarios**, respectively **falling-rate scenarios**.

STEP 2: ADD POSITIVITY, II.

Stressing **scenarios** is a new feature.

By a **rising-rate scenario** we mean:

- the specification of a fixed period of time $[t_0, T_0] \subseteq [0, T^*)$.
- the specification of a short-rate process $r = (r_u)_{u \in [0,T^*)}$ with $r|_{[t_0,T_0]}$ increasing w.r.t. the statistical measure P, i.e., $r_u \leq r_{u^*}$ on $(\Omega, \mathcal{F}, \mathbf{F}, P)$ for every $u \leq u^*$ in $[t_0, T_0]$.
- validity, for some $\varepsilon \geq 0$, of the P-almost-surely condition

$$(\mathrm{P}_\varepsilon) \quad r_u = f_{u,u} \geq \min\{0, f_{0,u} - \varepsilon\}, \text{ for every } u \in [t_0, T_0].$$

Falling-rate scenarios conceive as mirror images of rising-rate scenarios.

STEP 2: RISING-RATE SCENARIO, I.

- Will demonstrate incorporation of rising-rate scenarios
- in the situation when $L = \mathrm{IG}(\delta, \gamma)$-process
- in an as explicit and as strict as possible form of the no-arbitrage correspondence
- ... a plan which boils down to work done in the 2009/2010 MSc thesis of L. Slámová.

STEP 2: RISING-RATE SCENARIO, II.

- **Set-up** (§6.3 of Slámová MSc-thesis). Consider short rates $r = r(\sigma, (0, 0, \nu))$ of the form:

$$r_t = m_t + \int_0^t \sigma_{s,t} dL_s \,, \quad t \in \mathbf{R}_{\geq 0} \,,$$

where $L = L(0, 0, \nu)$ with

$$\nu(dx, ds) = \tfrac{1}{2}(1 + \gamma^2 x)\delta \frac{\exp\left(-\frac{1}{2}\gamma^2 x\right)}{\sqrt{2\pi x^3}} \mathbf{1}_{\{x>0\}} \, dx \times ds \,,$$

and where (generically)

$$m_t = f_{0,t} - \frac{\delta}{\gamma} \frac{\Sigma_{0,t}}{\sqrt{1+2\Sigma_{0,t}/\gamma^2}}$$

with $\Sigma_{0,t} = \int_0^t \sigma_{0,u} \, du = (\overline{\sigma}/a)(1 - \exp(-at))$.

STEP 2: RISING-RATE SCENARIO, III.

In the setting of the previous slide, the no-arbitrage correspondence with (MR) is explicitly given as follows.

- **Theorem 3** (Slámová MSc-thesis, §§6.3.2, 6.3.3): *For any two pairs* $(\overline{\sigma}, a) \in \mathbf{R}^2_{>0}$ *and* $(\delta, \gamma) \in \mathbf{R}^2_{\geq 0} \setminus \{(0,0)\}$, *the short rate* $r = r(\sigma, (0, 0, \nu))$ *constructed in the previous slide has the following properties* (1) *to* (3)*:*

  (1) r *satisfies* **No-arbitrage** *and* **FIT**.

  (2) *If* $f_{0,\infty} = \lim_{t\to\infty} f_{0,t}$ *exists in* $\mathbf{R}$, *then* r *has mean reversion to the random variable* r_∞ *given by:*

$$r_\infty = m_\infty + \mathrm{IG}(\delta_\infty, \gamma_\infty),$$

  where

$$m_\infty = f_{0,\infty} - \frac{\delta}{\gamma} \frac{\overline{\sigma}/a}{\sqrt{1-2(\overline{\sigma}/a)/\gamma^2}},$$

  and where $\delta_\infty = \delta\sqrt{\overline{\sigma}}/a$ *and* $\gamma_\infty = \gamma/\sqrt{\overline{\sigma}}$.

  (3) r *is increasing on* $[0, T^*)$ *for any* T^* *in* $[0, \infty]$.

STEP 2: RISING-RATE SCENARIO, IV.

The Theorem 3 explicit representation of the no-arbitrage correspondence gives a handle for establishing a **sufficiency criterion** for its compliance with an $\varepsilon = 0$ **rising-rate scenario**. This turns out to be based on **solving quadratic equations**.

- **Recall** for this

$$m_t = f_{0,t} - \frac{\delta}{\gamma} \frac{\Sigma_{0,t}}{\sqrt{1+2\Sigma_{0,t}/\gamma^2}}$$

  where $\Sigma_{0,t} = \int_0^t \sigma_{0,u}\, du = (\overline{\sigma}/a)(1 - \exp(-at))$.

- **The key equivalences:** Assuming $f_{0,t} > 0$,

  $0 \le m_t$ iff $\Sigma_{0,t}$ is in the solution interval of the quadratic inequality

$$Q_t(x) := \left(\frac{\delta}{\gamma}\right)^2 x^2 - 2\left(\frac{f_{0,t}}{\gamma}\right)^2 x - f_{0,t}^2 \le 0;$$

  iff $\overline{\sigma} \in \left[0, \frac{a}{1-\exp(-at)}\, \Sigma_t^+\right]$,

  where Σ_t^+ is the positive root of $Q_t(x) = 0$.

STEP 2: RISING-RATE SCENARIO, V.

Summing up, we have the following **criterion for compliance** of the short-rate process $r = r(\sigma, \mathrm{IG}(\delta, \gamma))$ constructed in Theorem 3 with an $\varepsilon = 0$ **rising-rate scenario**.

- **Theorem 4** (Slámová MSc-thesis, Corollary 6.26): *In the situation of* Theorem 3 *we have*

$$r_t > 0 \ \textit{ for any } t \in [t_0, T] \subseteq \mathbf{R}_{\geq 0} \cup \{\infty\}$$

if $f_{0,t} > 0$ *for all* $t \in [t_0, T]$ *and if the Vasicek vol parameter* $\overline{\sigma}$ *satisfies:*

$$\begin{aligned} \overline{\sigma} &\in \left(0, \min_{t \in [t_0, T]} \frac{a}{1 - \exp(-at)} \Sigma_t^+ \right) \\ &\subseteq \left(0, \frac{a}{1 - \exp(-at_0)} \min_{t \in [t_0, T]} \Sigma_t^+ \right), \end{aligned}$$

where $\Sigma_t^+ = (1/\delta^2)(f_{0,t}^2 + f_{0,t}\sqrt{f_{0,t}^2 + (\delta\gamma)^2}\ )$.

STEP 2: RISING-RATE SCENARIO, VI.

In the version of her MSc thesis prepared for Charles University, Prague, Slámová was able to develop:

- estimation techniques and
- simulation techniques

for rates r from the short rate machine including those of Theorems 3 and 4 above.

Examples of sample paths thus obtained include the following:

- **Figure 3:** Simulation of this talk's short rate r, based on drivers L constructed using $\mathrm{IG}(\delta, \gamma)$-processes.

Figure 3 by courtesy of Slámová.

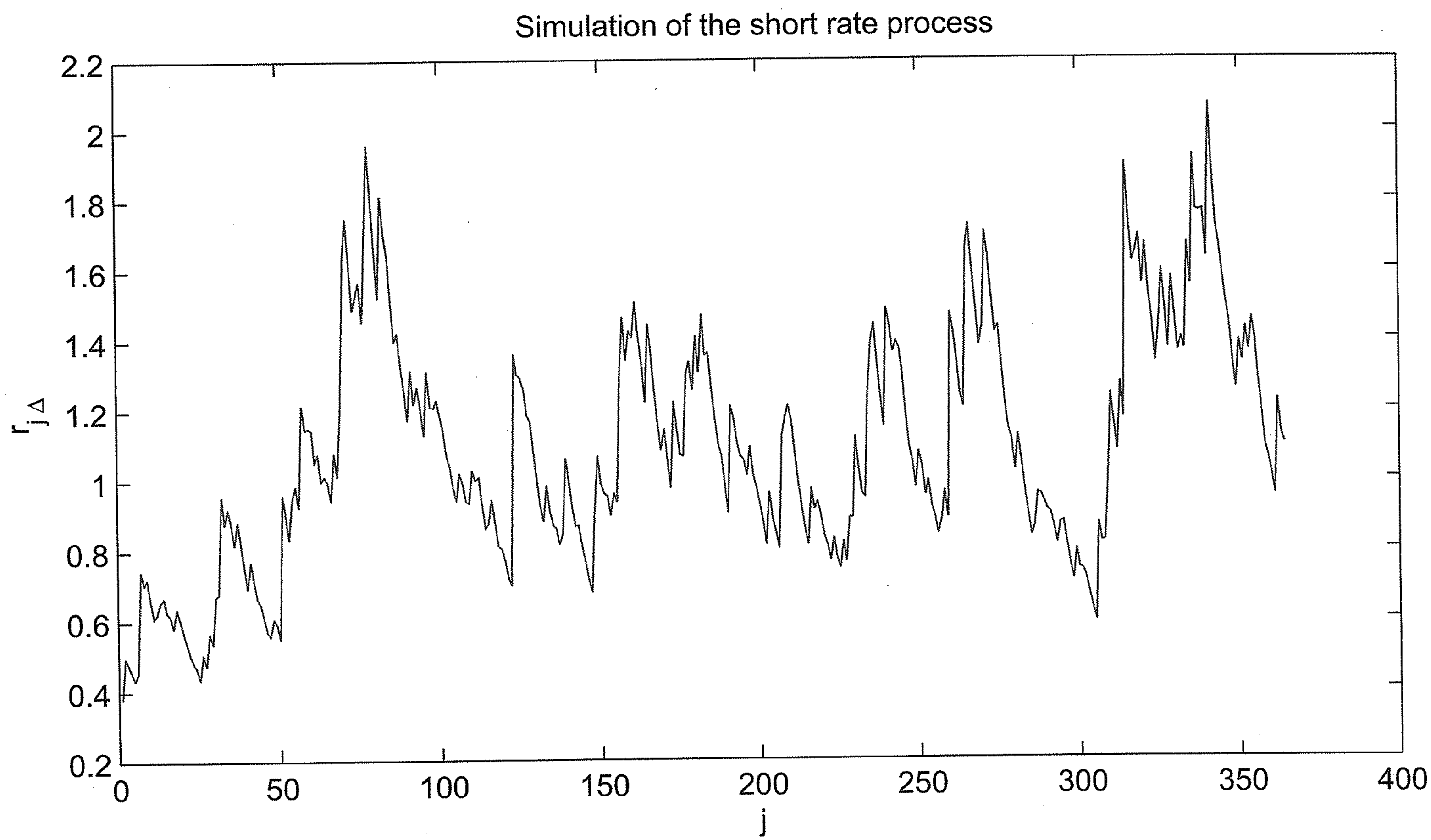


Figure 3. Simulation of r with $(\overline{\sigma}, a)=(0.1,\ 0.1)$ and $(\delta, \gamma)=(1,\ 10)$.

WHAT REMAINS TO BE ADDRESSED

(I) Application and use of this talk's approach and results to

- **exotics** valuation and hedging
- on **calibration** to liquid instruments.

As a small but necessary step in this direction:

- How to value plain vanilla options (e.g., on LIBOR)?

(II) How to dispose of the 'quadratic inequality' type restrictions in scenario-modelling.

VALUATION EXAMPLE, I.

- **Consider** the valuation of plain-vanilla call-options on time-T LIBOR with time-U payoff

$$(U-T)\,(\mathrm{LIBOR}_T-\kappa)_+\,.$$

- **Time-t value** of this call, using EMM Q, is given by

$$V_t = E^Q\big[B_t/B_U\,(U-T)\,(\mathrm{LIBOR}_T-\kappa)_+\,|\,\mathscr{F}_t\big]\,,$$

where $B_u = \exp\big(\int_{[0,u]} r_s\,ds\big)$.

- **Connect** with term-structure of interest-rates (as considered in this talk) by the 'equilibrium condition'

$$1+(U-T)\,\mathrm{LIBOR}_T = 1/P_{T,U}\,.$$

VALUATION EXAMPLE, II.

- **No-arbitrage dynamics** of bond values associated with a pair (σ, L) is explicitly given by

$$P_{T,U} = \frac{P_{0,U}}{P_{0,T}} \exp\big(- \int_0^T A_{s,T,U}\, ds + \int_0^T (-\Sigma_{s,T,U})\, dL_s\big)$$

where

$$\Sigma_{s,T,U} = \Sigma_{s,U} - \Sigma_{s,T} \text{ with } \Sigma_{s,u} = \int_0^u \sigma_{s,w}, dw\,,$$

$$A_{s,T,U} = \vartheta_{L,s}(-\Sigma_{s,U}) - \vartheta_{L,s}(-\Sigma_{s,T})\,.$$

- **Time-t value in no-arbitrage form** is given by

$$V_t = \kappa^* P_{t,T}\, E^{Q_T}\big[\big(c^* - \exp\big(- I_{t,T}(L)\big)\big)_+ \,|\, \mathscr{F}_t\big]\,,$$

computed w.r.t. the T-forward measure Q_T, where

$$I_{t,T}(L) = \int_{[t,T]} \Sigma_{s,T,U}\, dL_s\,,$$

and with normalized constants

$$\kappa^* = (1 + (U - T)\kappa) c_{t,T,U} \text{ and } c = 1/\kappa^*$$

where

$$c_{t,T,U} = (P_{0,U}/P_{0,T}) \exp\big(- \int_0^T A_{s,T,U}\, ds + \int_0^t (-\Sigma_{s,T,U})\, dL_s\big).$$

VALUATION EXAMPLE, III.

- **Idea:** Express V_t as a series in terms of higher derivatives of $\mathrm{MG}_L(z) = E^P[\exp(zL_1)]$.

- **Step 1 Reduction series** in terms of the integral order moments of $I_{t,T} = \int_{[t,T]} \Sigma_{s,T,U}\, dL_s$, typically:

$$V_t = \kappa^* P_{t,T} \sum_{n=0}^{\infty} a_n\, E^{Q_T}[L_n(I_{t,T})]\,,$$

where

$$L_n(x) = (n\text{-th Laguerre polynomial}) = \sum_{k=0}^{n} \alpha_{n,k} x^k$$

with $\alpha_{n,k} = (-1)^k/k!\binom{n}{k}$, and coefficients

$$a_n = \frac{\langle \phi, L_n \rangle}{\langle L_n, L_n \rangle} = \sum_{k=0}^{n} \alpha_{n,k} \int_0^\infty e^{-x} x^k \varphi(x)\, dx$$

exponential polynomials in $\log c^*$ on setting $\varphi(x) = (c^* - e^{-x})_+$.

- **Step 2 Application of the key lemma:** Obtain the moments of $I_{t,T}$ required in Step 1 in terms of the higher derivatives of MG_L by use (with $f(s) = z\Sigma_{s,T,U}$) of the **key lemma**:

$$E\big[\exp\big(\textstyle\int_{[t,T]} f(s)\, dL_s\big) \mid \mathscr{F}_t\big] = \exp\big(\textstyle\int_{[t,T]} \vartheta_{L,s}(f(s))\, ds\big)\,.$$

VALUATION EXAMPLE, IV.

- **Specialize** to the set-up of the **rising-rate scenario**:

$$\sigma_{s,u} = \overline{\sigma} \exp\left(-a(u-s)\right), \quad s \leq u,$$
$$L = \mathrm{IG}(\delta, \gamma),$$

and for the parameters choose the numerical values

$$\overline{\sigma} = 0.07 \quad a = 2.5$$
$$\delta = 0.1 \quad \gamma = 11.$$

(originally used to model a bulls' market in a stochastic volatility context ...).

- **Consider options** with $U - T = T = 6$ months and strike $\kappa = 0.75\%$ at time $t = 0$.
- **The value** V_t is then computed as

$$V_t = 0.99310695456.$$

VALUATION EXAMPLE, V.

- **Nota Bene:** Reduction series here once more enable accuracies of some 10 decimal places after the decimal point by merely using a single digit number of their terms; in the example some 5 terms are sufficient for this.

$$V_{t,0} = 0\,.\,93468704614\,5922415438268589589 0473$$
$$V_{t,1} = 0\,.\,93115233897\,427855429382671914 20976$$
$$V_{t,2} = 0\,.\,99310709401\,444078531252045808 11052$$
$$V_{t,3} = 0\,.\,99310695649\,460471654525926387 67005$$
$$V_{t,4} = 0\,.\,99310695459\,296486368176710832 20054$$
$$V_{t,5} = 0\,.\,99310695457\,016689432911213954 07522$$
$$V_{t,6} = 0\,.\,99310695456\,992153135501137233 09971$$

HOW TO DISPOSE OF (II)-TYPE RESTRICTIONS

Current approach is via **choice of EMMs** as follows.

- **Step 1.** Girsanov construction of **family of measures** $(Q_b)_{b \in \mathbf{R}}$ with each Q_b equivalent to P on $(\Omega, \mathbf{F})$. This construction preserves the principal form of the talk's no-arbitrage correspondence with (MR), as witnessed by the following 3 conditions for each Q_b.
  - Q_b gives rise to a no-arbitrage correspondence for forward-rate processes as in Theorem 1;i.e.,
  - Q_b becomes EMM on adaption of the drift of $(Q_b, \mathbf{F})$-dynamics of forward-rate process.
  - The resulting $(Q_b, \mathbf{F})$-dynamics of the short-rate process has (MR), under integrability conditions on (σ, L).
- **Step 2.** The realization of scenarios within each Step 1 no-arbitrage correspondence now rests solely on the choice of appropriate Q_b. Given (σ, L) and $\varepsilon \geq 0$ we show in particular
  - existence of **maximal** b_ε^* such that: the Theorem 4 type **rising-rate scenarios with** (P_ε) hold on $(\Omega, \mathcal{F}, \mathbf{F}, Q_b)$ for every $b \leq b_\varepsilon^*$.
- **Step 3.** Our methods for working with the models (e.g., reduction series) extend to the Step 1 and 2 framework.

HOPEFULLY TO BE CONTINUED

by further developing applications of this talk's approach to

- exotic fixed-income derivatives and their
- valuation and hedging.