Robustness, Model Ambiguity and Pricing

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Motivation

- Pricing contracts in incomplete markets
- Examples:
  - Pricing very long-dated cash flows $T \sim 30 - 100$ years
  - Pricing long-dated equity options $T > 5$ years
  - Pricing pension & insurance liabilities
- Actuarial premium principles typically "ignore" financial markets
  - Actuarial pricing is "static": price at $t = 0$ only
- Financial pricing considers "dynamic" pricing problem:
  - How does price evolve over time until time T?
- Financial pricing typically "ignores" unhedgeable risks

Outline of This Talk

1. Pricing in Complete Market
2. Robustness & Model Ambiguity
3. Applications
4. Summary & Conclusion

Tree Setup in Complete Market

Suppose we have a stock price S with return process $x = \ln S$:

$$dx = m\,dt + \sigma\,dW_x,$$

Discretisation in binomial tree:

$$x(t+\Delta t) = x(t) + \begin{cases} +\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 + \frac{m}{\sigma}\sqrt{\Delta t}) \\ -\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 - \frac{m}{\sigma}\sqrt{\Delta t}). \end{cases}$$

Model ambiguity as $m \in [m_L, m_H]$.

Valuation with Model Ambiguity

Suppose we have a derivative contract with value $f\big(t+\Delta t, x(t+\Delta t)\big)$ at time $t+\Delta t$.

Given uncertainty about drift m, “ambiguity averse” rational agent will consider “worst case” expectation:

$$\min_{m \in [m_L, m_H]} e^{-r\Delta t}\mathbb{E}_t^m[f\big(t+\Delta t, x(t+\Delta t)\big)]$$

Explicit solution for binomial tree:

$$\begin{cases} e^{-r\Delta t}\big(f_1 + (f_x m_L + \frac{1}{2} f_{xx}\sigma^2)\Delta t\big) \text{ if } f_x > 0 \\ e^{-r\Delta t}\big(f_1 + (\qquad\qquad \frac{1}{2} f_{xx}\sigma^2)\Delta t\big) \text{ if } f_x = 0 \\ e^{-r\Delta t}\big(f_1 + (f_x m_H + \frac{1}{2} f_{xx}\sigma^2)\Delta t\big) \text{ if } f_x < 0. \end{cases}$$

Interpretation of Valuation Equation

Take limit for $\Delta t \downarrow 0$.
Leads to semi-linear pde: $f_t + f_x m^* + \frac{1}{2} f_{xx} \sigma^2 - rf = 0$ with $m^* = m_L$ if $f_x > 0$ and $m^* = m_H$ if $f_x < 0$.

- Actuarial notion of *prudence* (not "risk-neutral")
- Time-consistent *coherent risk-measure* with "$\mathbb{Q} \in [m_L, m_H]$"
- *Good Deal Bound pricing* with upper bound on pricing kernel volatility
- GDB pricing with upper bound on Radon-Nikodym volatility

Model Ambiguity & Hedging

Suppose that rational agent can trade in the share price S.

Buy $\theta/S(t)$ shares at t, financed by borrowing an amount θ from the bank account B.

At time $t+\Delta t$, net position has value $(e^{x(t+\Delta t)-x(t)} - e^{r\Delta t})\theta$.

Find optimal amount θ that maximises worst-case expectation:

$$\max_{\theta} \min_{m\in[m_L,m_H]} e^{-r\Delta t}\big(f_1 + (f_x m + \tfrac{1}{2} f_{xx}\sigma^2 + (m + \tfrac{1}{2}\sigma^2 - r)\theta)\Delta t\big)$$

Two-player game: “mother nature” vs. agent.

Model Ambiguity & Hedging (2)

Optimum (m, θ) depends on sign of partial deriv's:

$$\frac{\partial}{\partial \theta} : e^{-r\Delta t}(m + \tfrac{1}{2}\sigma^2 - r)\Delta t \qquad \frac{\partial}{\partial m} : e^{-r\Delta t}(f_x + \theta)\sigma\Delta t$$

Optimal choice for m depends on sign of $\frac{\partial}{\partial m}$

- Suppose agent chooses θ such that $f_x + \theta > 0$,
- then "mother nature" chooses $m = m_L$.
- $\boxed{\text{If } m_L < r - \frac{1}{2}\sigma^2}$, then agent can improve by lowering θ,
- until $\theta = -f_x$.

- Similar argument for $f_x + \theta < 0$, $\boxed{\text{if } m_H > r - \frac{1}{2}\sigma^2}$

Model Ambiguity & Hedging (3)

Conclusion: optimal choice for agent is $\theta^* = -f_x$.

- But this is delta-hedge for derivative f
- Leads to risk-neutral valuation!

How severe is restriction $m_L < r - \frac{1}{2}\sigma^2$? (Equivalent to $\mu_L < r$)
Good Deal Bound should be higher than Market Price of Risk

Thought-experiment:

- Suppose 25 years of data
- $\hat{\mu} = 8\%$, $\sigma = 15\%$
- Then std.err. of estimate for $\hat{\mu}$ is $\sigma/\sqrt{25} = 15\%/5 = 3\%$
- So, 95%-conf.intv. for $\hat{\mu}$ is $8\% \pm 6\%$.
- Need about $(2 \cdot 15/(8-4))^2 \approx 50$ years of data to distinguish between 8% and 4% if $\sigma = 15\%$!

Tree Setup for Incomplete Market

Remember we have a stock price S with return process $x = \ln S$:

$$dx = m\,dt + \sigma\,dW_x,$$

Discretisation in binomial tree:

$$x(t+\Delta t) = x(t) + \begin{cases} +\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 + \frac{m}{\sigma}\sqrt{\Delta t}) \\ -\sigma\sqrt{\Delta t} & \text{with prob. } \frac{1}{2}(1 - \frac{m}{\sigma}\sqrt{\Delta t}). \end{cases}$$

Model ambiguity as $m \in [m_L, m_H]$.

- Change mean $\iff$ change probability $\iff$ stoch. discount factor
- "Local Volatility" of stoch. discount factor: $m/\sigma\sqrt{\Delta t}$
- Conf.Intv. on mean $\iff$ Good Deal Bounds on discount factor vola
- Indistinguishable models $\iff$ Likelihood ratio test

Tree Setup (2)

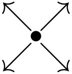
Introduce additional non-traded process y:

$$dy = a\,dt + b\,dW_y,$$

with $dW_x\,dW_y = \rho\,dt$.

"Quadrinomial" discretisation:

State:	$y + b\sqrt{\Delta t}$	$y - b\sqrt{\Delta t}$
$x + \sigma\sqrt{\Delta t}$	$p_{++} = \left(\frac{(1+\rho)+(\frac{m}{\sigma}+\frac{a}{b})\sqrt{\Delta t}}{4}\right)$	$p_{+-} = \left(\frac{(1-\rho)+(\frac{m}{\sigma}-\frac{a}{b})\sqrt{\Delta t}}{4}\right)$
$x - \sigma\sqrt{\Delta t}$	$p_{-+} = \left(\frac{(1-\rho)-(\frac{m}{\sigma}-\frac{a}{b})\sqrt{\Delta t}}{4}\right)$	$p_{--} = \left(\frac{(1+\rho)-(\frac{m}{\sigma}+\frac{a}{b})\sqrt{\Delta t}}{4}\right)$



Model Ambiguity

Uncertainty in both parameters m and a.

Additional notation:

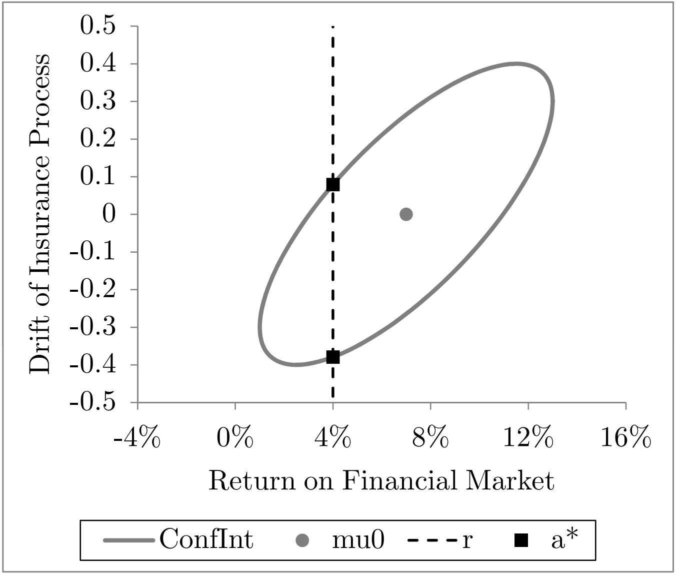
$$\mu := \begin{pmatrix} m \\ a \end{pmatrix}, \quad \Sigma := \begin{pmatrix} \sigma^2 & \rho\sigma b \\ \rho\sigma b & b^2 \end{pmatrix}.$$

Describe ambiguity set as ellipsoid:

$$\mathcal{K} := \{\mu_0 + \varepsilon \mid \varepsilon' \Sigma^{-1} \varepsilon \leq k^2\}.$$

- Motivated by shape of confidence interval of estimator $\hat{\mu}$
- Motivated by Good Deal Bound
- Motivated by Likelihood Ratio Testing

Ellipsoid Ambiguity Set



Robust Optimisation Problem

Consider derivative f with payoff $f\big(t+\Delta t, x(), y()\big)$ at time $t+\Delta t$.
Consider hedged position: $f(t+\Delta t) + \theta\big(e^{x(t+\Delta t)-x(t)} - e^{r\Delta t}\big)$

Ambiguity averse rational agent solves the following optimisation problem for a time-step Δt:

$$\max_{\theta} \min_{\mu\in\mathcal{K}} e^{-r\Delta t}\Big(f_1 + \big(\nabla f'\mu + \theta(e_1'\mu - r + \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}\operatorname{tr}(f_{xx}\Sigma)\big)\Delta t\Big),$$

where ∇f denotes gradient $(f_x, f_y)'$ and e_1 denotes the vector $(1,0)'$.

Reformulate & simplify problem

$$\begin{aligned} \max_{\theta} \min_{\varepsilon} \quad & \theta q + \varepsilon'(\nabla f + \theta e_1) \\ \text{s.t.} \quad & \varepsilon'\Sigma^{-1}\varepsilon \le k^2. \end{aligned}$$

with $q = (e_1'\mu_0 - r + \frac{1}{2}\sigma^2)$ is excess return

Optimal Response for Mother Nature

Two-player game: agent vs. "Mother Nature"

Worst-case choice for Mother Nature given any θ is "opposite direction" of vector $(\nabla f + \theta e_1)$:

$$\varepsilon^* := -\left(\frac{k}{\sqrt{(\nabla f + \theta e_1)' \Sigma (\nabla f + \theta e_1)}} \right) \Sigma (\nabla f + \theta e_1).$$

If we use this value for ε^* we obtain the reduced optimisation problem for the agent:

$$\max_{\theta} \quad \theta q - k \sqrt{(\nabla f + \theta e_1)' \Sigma (\nabla f + \theta e_1)}.$$

Maximise expected excess return θq minus k times st.dev. of portfolio. Similar to maximise w.r.t. Cost-of-Capital "penalty".

Optimal Response for Agent

Solution to reduced optimisation problem for agent:

$$\theta^* := -\left(f_x + \frac{b\rho}{\sigma} f_y\right) + \frac{q/\sigma}{\sqrt{k^2 - (q/\sigma)^2}} \frac{b\sqrt{1-\rho^2}}{\sigma} |f_y|.$$

Nice economic interpretation:

- Left term is best possible hedge
  - Perfect hedge for "pure financial" risks
  - Induces market-consistent pricing
- Right term is "speculative" position, which is product of:
  - Function of Sharpe ratio q/σ
  - Residual unhedgeable risk
  - Absolute value of f_y

Agent's Valuation of Contract

If we substitute optimal ε^* and θ^* into original expectation, we obtain semi-linear pde

$$f_t + f_x(r - \tfrac{1}{2}\sigma^2) + f_y a^* + \tfrac{1}{2}\sigma^2 f_{xx} + \rho\sigma b f_{xy} + \tfrac{1}{2} b^2 f_{yy} - rf = 0,$$

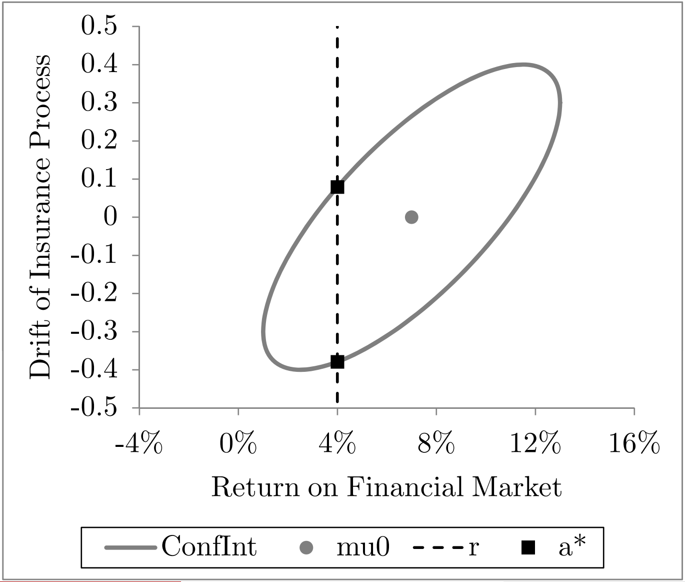
where the drift term a^* for the insurance process is given by

$$a^* = \left(a_0 - \frac{\rho b}{\sigma} q\right) \mp \left(\sqrt{k^2 - \left(\tfrac{q}{\sigma}\right)^2}\right) b\sqrt{1-\rho^2},$$

where $\mp$ depends on sign of f_y.

Again, nice economic interpretation for a^*.
Same result as *Good Deal Bound* pricing.
Interpretation as *Cost-of-Capital* pricing from insurance industry.

Agent's Valuation of Contract – Graphical



Different Interpretations for k

- Equivalence between Good Deal Bound & Model Ambiguity
- Parameter k is:
  - Width of confidence interval for trend
  - Volatility of pricing kernel; stoch.disc.factor
  - Sharpe-ratio of risks
  - Cost-of-Capital times # of st.dev's for unhedgeable risk
- Calculation of k:
  - Sharpe-ratio for equity: $k > (8\% - 4\%)/16\% = 0.25$
  - Conf.intv.: $k \approx 2/\sqrt{25} = 0.4$
  - Cost-of-Cap: $k \approx 0.06 * 2.5 = 0.15$: too low?
- It seems reasonable to set $k \approx 0.3$.

Applications

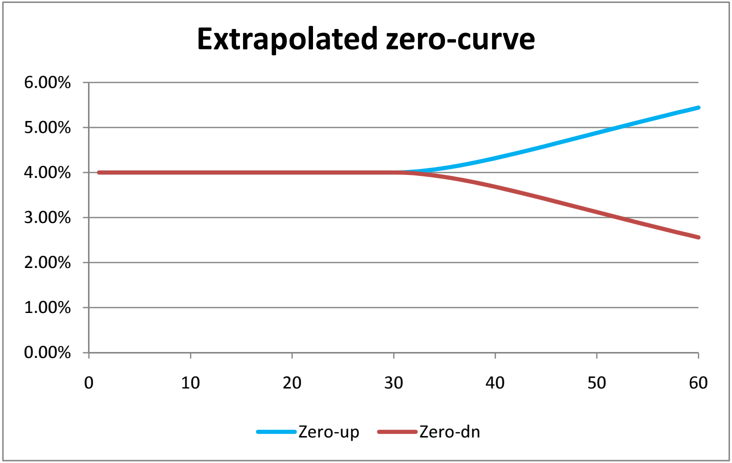
- Pricing Very Long-Dated Cash Flows
- Pricing Longevity Risk
- Pricing Unhedgeable Equity Risk

Pricing Very Long-Dated Cash Flows

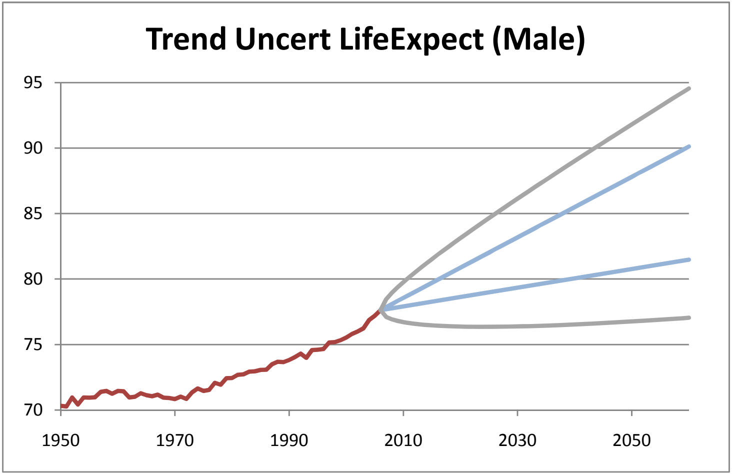
- Life Insurance and Pension cash flows extend to 70 years
- Market for Government Bonds extends to only 30 years
- Market for Discount Bonds incomplete beyond 30 years

- Use term-structure for interest rate up to 30 years
- After 30 years use "robust pricing"
- Example:
  - Assume 1-factor Hull-White model
  - $\sigma_{HW} = 0.01, a_{HW} = 0.05$, long term rate: $z_\infty = 4\%$
  - Take $k = 0.3$, then price at LT rate of $z_\infty - k/a\,\sigma_{HW} = -2\%$.
  - Mean-rev determines transition between z_{30} and $z_\infty - k/a\,\sigma_{HW}$

Extrapolation of Zero-curve



Trend Uncertainty Life Expectancy



Life Expectancy (at birth) for Dutch Males 1950 until 2006
Conf.intv. for trend: $[0.9, 2.8]$ months per annum.

Pricing Longevity Risk

- Best Estimate trend for increase in life expectancy is 1.8 months per annum
- Standard Deviation of process is 3.7 months per annum
- Robust Approach:
  - Price long life risk at trend of 3.7 months p/a
  - Price short life risk at trend of 0.0 months p/a
  - Combined portfolio: price at “net exposure”

Summary & Conclusion

- Robust agent holds hedge portfolio + speculative position
- Price contracts in incomplete markets in a “market-consistent” way
- Robust agent prices unhedgeable risks using a “worst case” drift

Connections to:

- Actuarial notion of *prudence*
- *Good Deal Bound* pricing (see Cochrane & Saa-Requejo)
- *Confidence Interval* for drift parameters
- *Likelihood Ratio Testing* of models (see Hansen & Sargent)
- *Cost-of-Capital* pricing used by industry (see QIS5)